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Expansion formulas for the inertias of Hermitian matrix polynomials and matrix pencils of orthogonal projectors

Yongge Tian

China Economics and Management Academy, Central University of Finance and Economics, Beijing 100081, China

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ABSTRACT

This paper gives a group of expansion formulas for the inertias of Hermitian matrix polynomials $A - A^2$, $I - A^2$ and $A - A^3$ through some congruence transformations for block matrices, where A is a Hermitian matrix. Then, the paper derives various expansion formulas for the ranks and inertias of some matrix pencils generated from two or three orthogonal projectors and Hermitian unitary matrices. As applications, the paper establishes necessary and sufficient conditions for many matrix equalities to hold, as well as many inequalities in the Löwner partial ordering to hold.

1. Introduction

Throughout this paper, $\mathbb{C}^{m\times n}$ and $\mathbb{C}_{\mathrm{H}}^{m}$ denote the collections of all $m \times n$ complex matrices and all $m \times m$ complex Hermitian matrices, respectively. The symbols A^*, $r(A)$ and $\mathscr{R}(A)$ stand for the conjugate transpose, rank, range (column space) of a matrix $A \in \mathbb{C}^{m\times n}$, respectively; I_m denotes the identity matrix of order m; $[A, B]$ denotes a row block matrix consisting of A and B. Two Hermitian matrices A and B of the same size are said to be congruent if there is an invertible matrix S such that $SAS^* = B$. We write $A > 0$ ($A \geqslant 0$) if A is Hermitian positive (nonnegative) definite. Two Hermitian matrices A and B of the same size are said to satisfy the inequality $A > B$ ($A \geqslant B$) in the Löwner partial ordering if $A - B$ is positive (nonnegative) definite; cf. Löwner [35, p. 177], and also Marshall and Olkin [37, p. 462]. The Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the four matrix equations $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$. In particular, $a^\dagger = a^{-1}$ if $a \neq 0$ and $a^\dagger = 0$ if $a = 0$ for a scalar $a \in \mathbb{C}$. Some well-known equalities for the Moore–Penrose inverse are given by

$$A^\dagger = A^*\left(AA^*\right)^\dagger = \left(A^*A\right)^\dagger A^* = A^*\left(A^*AA^*\right)^\dagger A^*, \qquad A^* = A^\dagger AA^* = A^*AA^\dagger; \tag{1.1}$$

see [62]. Results on the Moore–Penrose inverse can be found, e.g., in [12,13,29].

A matrix $A \in \mathbb{C}^{m\times m}$ is called an orthogonal projector if it is both idempotent and Hermitian, i.e., $A^2 = A = A^*$; the collection of all orthogonal projectors of order m is denoted by $\mathbb{C}_{\mathrm{OP}}^{m}$. A matrix $A \in \mathbb{C}^{m\times m}$ is said to be Hermitian unitary if $A = A^* = A^{-1}$, and the collection of all Hermitian unitary matrices of order m is denoted by $\mathbb{C}_{\mathrm{HU}}^{m}$. A matrix $X \in \mathbb{C}^{m\times m}$ is called the orthogonal projector onto the range $\mathscr{R}(A)$ of $A \in \mathbb{C}^{m\times n}$, denoted by $X = P_A$, if it satisfies $\mathscr{R}(X) = \mathscr{R}(A)$ and $X^2 = X = X^*$. It can be seen from the definition of the Moore–Penrose inverse that the orthogonal projector onto

E-mail address: yongge.tian@gmail.com.

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$\mathscr{R}(A)$ can uniquely be represented as $P_A = AA^\dagger$. Further, denote $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$, both of which are orthogonal projectors onto the null spaces of A^* and A, respectively, and their ranks are given by $r(E_A) = m - r(A)$ and $r(F_A) = n - r(A)$.

When considering a Hermitian matrix, we are usually concerned with distributions of the eigenvalues of the matrix, as well as its definiteness. Recall that the eigenvalues of a Hermitian matrix $A \in \mathbb{C}_{\mathrm{H}}^m$ are all real numbers, and the inertia of A is defined to be the triplet

$$\mathrm{In}(A) = \big\{i_+(A), i_-(A), i_0(A)\big\},$$

where $i_+(A)$, $i_-(A)$ and $i_0(A)$ are the numbers of the positive, negative and zero eigenvalues of A counted with multiplicities, respectively. The two numbers $i_+(A)$ and $i_-(A)$ are usually called the partial inertia of A; see, e.g., [9]. The difference $i_+(A) - i_-(A)$, denoted by $s(A)$, is usually called the signature of A. For a matrix $A \in \mathbb{C}_{\mathrm{H}}^m$, we have $r(A) = i_+(A) + i_-(A)$ and $i_0(A) = m - r(A)$. Hence, once the partial inertia $i_\pm(A)$ is determined, $r(A)$, $i_0(A)$ and $s(A)$ are obtained as well.

This paper aims at establishing some basic formulas for inertias of certain simple polynomials consisting of a Hermitian matrix, and then using the formulas to derive a variety of equalities for ranks/inertias of various matrix expressions consisting of orthogonal projectors. As applications, the author gives necessary and sufficient conditions for a wealth of matrix equalities and inequalities consisting of orthogonal projectors to hold.

Note that the inertia of a Hermitian matrix describes the sign distribution of the real eigenvalues of the matrix. Hence, it can be used to characterize definiteness of the matrix. The following results are obvious from the definitions of the rank/inertia of a matrix.

Lemma 1.1. *Let $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times n}$, and $C \in \mathbb{C}_{\mathrm{H}}^m$. Then,*

(a) *A is nonsingular if and only if $r(A) = m$.*
(b) *$B = 0$ if and only if $r(B) = 0$.*
(c) *$C > 0$ ($C < 0$) if and only if $i_+(C) = m$ ($i_-(C) = m$).*
(d) *$C \geqslant 0$ ($C \leqslant 0$) if and only if $i_-(C) = 0$ ($i_+(C) = 0$).*

This lemma shows that once certain formulas for ranks/inertias of Hermitian matrices and their operations are derived, we can use them to characterize equalities and inequalities for matrices. This basic algebraic method, which we refer to as the matrix rank/inertia method, is quite effective to solve various problems on conditional matrix equalities and inequalities in matrix theory and applications. It is well known in undergraduate linear algebra course that a direct method for computing the inertia of a Hermitian matrix is to reduce the matrix to a diagonal form by congruence transformations. This method is unstable for computing the exact inertia of a general matrix from the numerical point of view, so that computing the inertia of a matrix is regarded as a hard problem in linear algebra and no method is known to get the inertia of a general matrix exactly; see, e.g., [27,28]. From the symbolical point of view, the congruence transformation is the only method to study algebraic properties of Hermitian matrices. Without much effort, many closed-form formulas for ranks/inertias of Hermitian matrices and their operation can be established through congruence transformations; see the author's recent papers [50,51,54], and the results in the sections below.

We shall repeatedly use the simple or well-known results on ranks and inertias of (Hermitian) matrices in the following lemmas.

Lemma 1.2. *(See [36, Theorem 19].) Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$ be given. Then,*

$$r[A, B] = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{1.2}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C). \tag{1.3}$$

In particular,

$$r[A, B] = r(A) + r(B) \quad \Leftrightarrow \quad \mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}. \tag{1.4}$$

Lemma 1.3. *Let $A \in \mathbb{C}_{\mathrm{H}}^m$, $B \in \mathbb{C}_{\mathrm{H}}^n$, $Q \in \mathbb{C}^{m\times n}$, and assume that $P \in \mathbb{C}^{m\times m}$ is nonsingular. Then,*

$$i_\pm\big(PAP^*\big) = i_\pm(A), \tag{1.5}$$

$$i_\pm\big(A^{2k-1}\big) = i_\pm(A) \quad \textit{and} \quad i_\pm\big(A^\dagger\big) = i_\pm(A) \quad \textit{for any integer } k \geqslant 1, \tag{1.6}$$

$$i_\pm(\lambda A) = \begin{cases} i_\pm(A) & \textit{if } \lambda > 0, \\ i_\mp(A) & \textit{if } \lambda < 0, \end{cases} \tag{1.7}$$

$$i_{\pm}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = i_{\pm}(A) + i_{\pm}(B), \tag{1.8}$$

$$i_{\pm}\begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = r(Q). \tag{1.9}$$

Eq. (1.5) is the well-known Sylvester's law of inertia, which was first established in 1852 by Sylvester [45] (see also [30, Theorem 4.5.8] and [38, p. 377]). Eq. (1.6) follows from the fact that the signs of nonzero eigenvalues of A, A^{2k-1} and $A^{\dagger}$ are the same. Eqs. (1.7) and (1.8) are obvious from the definition of inertia, and (1.9) is well known; see, e.g., [25,26].

We also need the following results on ranks/inertias of Hermitian matrices.

Lemma 1.4. *Let $A, B \in \mathbb{C}_{\mathrm{H}}^{m}$, and assume that $AB = BA$. Then,*

(a) *Both $A^{2k-1}B$ and AB^{2k-1} are Hermitian, and*

$$i_{\pm}\big(A^{2k-1}B\big) = i_{\pm}\big(AB^{2k-1}\big) = i_{\pm}(AB) \tag{1.10}$$

for any integer $k \geqslant 1$.

(b) *If $B \geqslant 0$, then $A^{2k}B \geqslant 0$, and*

$$i_{+}\big(A^{2k}B\big) = r\big(A^{2k}B\big) = r(AB) \tag{1.11}$$

for any integer $k \geqslant 1$.

Proof. It is well known that under the conditions $A = A^*$, $B = B^*$ and $AB = BA$, there exists a unitary matrix U such that $A = U\,\mathrm{diag}\{\lambda_1, \ldots, \lambda_m\}U^*$ and $B = U\,\mathrm{diag}\{\mu_1, \ldots, \mu_m\}U^*$, where $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_m$ are the real eigenvalues of A and B, respectively. In this case,

$$AB = U\,\mathrm{diag}\{\lambda_1\mu_1, \ldots, \lambda_m\mu_m\}U^*,$$
$$A^{2k-1}B = U\,\mathrm{diag}\big\{\lambda_1^{2k-1}\mu_1, \ldots, \lambda_m^{2k-1}\mu_m\big\}U^*, \qquad AB^{2k-1} = U\,\mathrm{diag}\big\{\lambda_1\mu_1^{2k-1}, \ldots, \lambda_m\mu_m^{2k-1}\big\}U^*,$$

where $\lambda_i\mu_i$, $\lambda_i^{2k-1}\mu_i$ and $\lambda_i\mu_i^{2k-1}$ have the same sign, $i = 1, \ldots, m$. Hence, (1.10) follows. Under the condition $B \geqslant 0$, we have $A^{2k}B = U\,\mathrm{diag}\{\lambda_1^{2k}\mu_1, \ldots, \lambda_m^{2k}\mu_m\}U^* \geqslant 0$. Thus (1.11) follows. $\square$

Lemma 1.5. *(See [50, Theorem 2.3].) Let $A \in \mathbb{C}_{\mathrm{H}}^{m}$, $B \in \mathbb{C}^{m\times n}$, $D \in \mathbb{C}_{\mathrm{H}}^{n}$, and denote*

$$M_1 = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}.$$

Then,

$$i_{\pm}(M_1) = r(B) + i_{\pm}(E_BAE_B), \tag{1.12}$$

$$r(M_1) = 2r(B) + r(E_BAE_B), \tag{1.13}$$

$$i_{\pm}(M_2) = i_{\pm}(A) + i_{\pm}\begin{bmatrix} 0 & E_AB \\ B^*E_A & D - B^*A^{\dagger}B \end{bmatrix}, \tag{1.14}$$

$$r(M_2) = r(A) + r\begin{bmatrix} 0 & E_AB \\ B^*E_A & D - B^*A^{\dagger}B \end{bmatrix}. \tag{1.15}$$

In particular,

(a) *If $A \geqslant 0$, then*

$$i_{+}(M_1) = r[A, B], \qquad i_{-}(M_1) = r(B), \qquad r(M_1) = r[A, B] + r(B). \tag{1.16}$$

(b) *If $A \leqslant 0$, then*

$$i_{+}(M_1) = r(B), \qquad i_{-}(M_1) = r[A, B], \qquad r(M_1) = r[A, B] + r(B). \tag{1.17}$$

(c) *If $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, then*

$$i_{\pm}(M_2) = i_{\pm}(A) + i_{\pm}\big(D - B^*A^{\dagger}B\big), \qquad r(M_2) = r(A) + r\big(D - B^*A^{\dagger}B\big). \tag{1.18}$$

(d) *If* $\mathscr{R}(B) \cap \mathscr{R}(A) = \{0\}$ *and* $\mathscr{R}(B^*) \cap \mathscr{R}(D) = \{0\}$, *then*

$$i_{\pm}(M_2) = i_{\pm}(A) + i_{\pm}(D) + r(B), \qquad r(M_2) = r(A) + 2r(B) + r(D). \tag{1.19}$$

In order to simplify block matrices, we adopt the following three types of elementary block matrix operations (EBMOs, for short): (I) interchange two block rows (columns) in a block matrix; (II) multiply a block row (column) by a nonsingular matrix from the left-hand (right-hand) side in a block matrix; (III) add a block row (column) multiplied by a matrix from the left-hand (right-hand) side to another block row (column). It is obvious that EBMOs don't change the rank of a block matrix.

The rest of this paper is organized as follows. In Section 2, we first construct a group of congruence transformations for some block matrices consisting of a Hermitian matrix A and its operations. From the congruence transformations and the Sylvester's law of inertia, we derive some basic formulas for the partial inertias of the matrix polynomials $A - A^2$, $I_m - A^2$ and $A - A^3$ in terms of the partial inertias of A and $I_m \pm A$, and present some variations of the expansion formulas. In Sections 3 and 4, we give a variety of expansion formulas for the partial inertias of matrix pencils generated from two or more orthogonal projectors and their operations, and present various consequences and applications of these formulas. Section 5 gives some expansion formulas for the inertias of orthogonal projectors onto the ranges of block matrices and submatrices. Section 6 gives expansion formulas for partial inertias of some matrix pencils generated from Hermitian unitary matrices. Section 7 proposes some problems on inertias of Hermitian matrices for further consideration.

2. Expansion formulas for ranks/inertias of some polynomials of a Hermitian matrix

When considering a quantity in mathematics, it is always desirable to establish some informative expansion formulas for the quantity. Once certain expansion formulas for the quantity are established, we can use them to derive various properties of the quantity. This is a quite inclusive but challenging topic in mathematics and applications. In matrix theory, many numerical characteristics of matrices can be defined, and of course, some valuable expansion formulas for such numerical characteristics are expected to establish. For a given Hermitian matrix, one of the most basic concepts associated with the matrix is its inertia. In a recent paper [50], the present author collected some well-known formulas for inertias of Hermitian matrices, and also showed many new expansion formulas for inertias of block Hermitian matrices, products of Hermitian matrices, and sums of Hermitian matrices. In particular, the present author gave some expansion formulas for the inertias of $A \pm B$, where A and B are both Hermitian matrices. As a continuation, we derive expansion formulas for the ranks/inertias of the three matrix polynomials $A - A^2$, $I_m - A^2$ and $A - A^3$, where A is a Hermitian matrix of order m.

For any given square matrix A of order m, the following three simple and interesting rank formulas

$$r\big(A - A^2\big) = r(A) + r(I_m - A) - m, \tag{2.1}$$

$$r\big(I_m - A^2\big) = r(I_m + A) + r(I_m - A) - m, \tag{2.2}$$

$$r\big(A - A^3\big) = r(A) + r(I_m + A) + r(I_m - A) - 2m \tag{2.3}$$

are well known in undergraduate linear algebra. These rank formulas can be proved in about a paragraph by using only an idea of elementary matrix operations to certain block matrices consisting of I_m, A and their operations. For example, (2.1) can be derived from the following two-sided elementary block matrix operations

$$\begin{bmatrix} I_m & 0 \\ -A & I_m \end{bmatrix} \begin{bmatrix} I_m & I_m - A \\ A & 0 \end{bmatrix} \begin{bmatrix} I_m & -I_m + A \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & A^2 - A \end{bmatrix}, \tag{2.4}$$

$$\begin{bmatrix} I_m & -I_m \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_m & I_m - A \\ A & 0 \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -I_m & I_m \end{bmatrix} = \begin{bmatrix} 0 & I_m - A \\ A & 0 \end{bmatrix}. \tag{2.5}$$

More elementary block matrix transformations and the corresponding rank equalities for matrix polynomials can be found in the literature; see, e.g., [1,55,57]. The three elementary formulas in (2.1)–(2.3) show such an interesting fact that the ranks of the three matrix polynomials $A - A^2$, $I_m - A^2$ and $A - A^3$ can be calculated through the algebraic operations of the ranks of their multiplication factors A and $I_m \pm A$. Hence, (2.1)–(2.3) can be called as expansion formulas for the ranks of the three matrix polynomials. Eqs. (2.1)–(2.3) can be used to characterize some basic algebraic properties of $A - A^2$, $I_m - A^2$ and $A - A^3$, such as, the nonsingularity of three matrix polynomials, as well as the idempotency, involution and tripotency of A.

The rank expansion formulas in (2.1)–(2.3) hold, of course, for any Hermitian matrix A of order m. Also, recall that the rank of a Hermitian matrix A is the sum of the partial inertia of A. Therefore, the rank and inertia of a Hermitian matrix should share the same mechanism. In fact, the expansion formulas in (1.12)–(1.15) show such reasonable separations between the ranks and partial inertias of block Hermitian matrices. This fact prompts us to consider some reasonable separations of (2.1)–(2.3) into certain expansion formulas for the partial inertias of $A - A^2$, $I_m - A^2$ and $A - A^3$. The approach pursued in this section is in a similar spirit to (2.4) and (2.5), but relies primarily on congruence transformations for some block Hermitian matrices consisting of I_m, A and their operations, as well as the formulas in Lemma 1.3.

Theorem 2.1. *Let $A \in \mathbb{C}_{\mathrm{H}}^{m}$. Then,*

$$i_+\big(A - A^2\big) = i_+(A) + i_+(I_m - A) - m, \tag{2.6}$$

$$i_-\big(A - A^2\big) = i_-(A) + i_-(I_m - A), \tag{2.7}$$

$$s\big(A - A^2\big) = s(A) + s(I_m - A) - m, \tag{2.8}$$

$$i_+\big(I_m - A^2\big) = i_+(I_m + A) + i_+(I_m - A) - m, \tag{2.9}$$

$$i_-\big(I_m - A^2\big) = i_-(I_m + A) + i_-(I_m - A), \tag{2.10}$$

$$s\big(I_m - A^2\big) = s(I_m + A) + s(I_m - A) - m, \tag{2.11}$$

$$i_\pm\big(A - A^3\big) = i_\pm(A) + i_\mp(I_m + A) + i_\pm(I_m - A) - m, \tag{2.12}$$

$$s\big(A - A^3\big) = s(A) + s(I_m - A) - s(I_m + A). \tag{2.13}$$

Hence,

(a) *$A - A^2 > 0$ ($A - A^2 \geqslant 0$) if and only if $I_m > A > 0$ ($I_m \geqslant A \geqslant 0$), i.e., A is a strict contraction (contraction).*
(b) *$A - A^2 < 0$ ($A - A^2 \leqslant 0$) if and only if $i_-(I_m - A) + i_-(A) = m$ ($i_+(I_m - A) + i_+(A) = m$).*
(c) *$I_m - A^2 > 0$ ($I_m - A^2 \geqslant 0$) if and only if $I_m > A > -I_m$ ($I_m \geqslant A \geqslant -I_m$).*
(d) *$I_m - A^2 < 0$ ($I_m - A^2 \leqslant 0$) if and only if $i_-(I_m + A) + i_-(I_m - A) = m$ ($i_+(I_m + A) + i_+(I_m - A) = m$).*
(e) *$A - A^3 > 0$ ($A - A^3 \geqslant 0$) if and only if $i_-(I_m + A) + i_+(I_m - A) + i_+(A) = 2m$ ($i_+(I_m + A) + i_-(I_m - A) + i_-(A) = m$).*
(f) *$A - A^3 < 0$ ($A - A^3 \leqslant 0$) if and only if $i_+(I_m + A) + i_-(I_m - A) + i_-(A) = 2m$ ($i_-(I_m + A) + i_+(I_m - A) + i_+(A) = m$).*

Proof. It is easily seen from (1.5) that if

$$P_1 M P_1^* = N_1 \quad \text{and} \quad P_2 M P_2^* = N_2 \tag{2.14}$$

for three Hermitian matrices M, N_1 and N_2 of the same size, where P_1 and P_2 are both nonsingular, then

$$i_\pm(M) = i_\pm(N_1) = i_\pm(N_2). \tag{2.15}$$

Now let

$$M_1 = \begin{bmatrix} 2^{-1}I_m & 2^{-1}I_m - A \\ 2^{-1}I_m - A & 2^{-1}I_m \end{bmatrix}, \qquad M_2 = \begin{bmatrix} I_m & A \\ A & I_m \end{bmatrix}, \qquad M_3 = \begin{bmatrix} -A & 0 & I_m \\ 0 & A & A \\ I_m & A & 0 \end{bmatrix}. \tag{2.16}$$

Then, the three block matrices are all Hermitian. Also, it is easily verified that

$$P_1 M_1 P_1^* = 2\begin{bmatrix} I_m - A & 0 \\ 0 & A \end{bmatrix}, \qquad P_1 = \begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}, \tag{2.17}$$

$$Q_1 M_1 Q_1^* = \begin{bmatrix} 2^{-1}I_m & 0 \\ 0 & 2(A - A^2) \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} I_m & 0 \\ -I_m + 2A & I_m \end{bmatrix}, \tag{2.18}$$

$$P_2 M_2 P_2^* = 2\begin{bmatrix} I_m + A & 0 \\ 0 & I_m - A \end{bmatrix}, \qquad P_2 = \begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}, \tag{2.19}$$

$$Q_2 M_2 Q_2^* = \begin{bmatrix} I_m & 0 \\ 0 & I_m - A^2 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} I_m & 0 \\ -A & I_m \end{bmatrix}, \tag{2.20}$$

$$P_3 M_3 P_3^* = \begin{bmatrix} I_m - A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -I_m - A \end{bmatrix}, \qquad P_3 = \begin{bmatrix} \frac{1}{\sqrt{2}}I_m & -\frac{1}{\sqrt{2}}I_m & \frac{1}{\sqrt{2}}I_m \\ 0 & I_m & 0 \\ \frac{1}{\sqrt{2}}I_m & \frac{1}{\sqrt{2}}I_m & -\frac{1}{\sqrt{2}}I_m \end{bmatrix}, \tag{2.21}$$

$$Q_3 M_3 Q_3^* = \begin{bmatrix} 0 & 0 & I_m \\ 0 & A - A^3 & 0 \\ I_m & 0 & 0 \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} I_m & 0 & 2^{-1}A \\ -A & I_m & -A^2 \\ 0 & 0 & I_m \end{bmatrix}, \tag{2.22}$$

and that the six block matrices P_i and Q_i, $i = 1, 2, 3$, are all nonsingular. Applying (2.15) to (2.17)–(2.22) and simplifying by (1.7)–(1.9), we obtain the following equalities for the partial inertias of M_1, M_2 and M_3

$$i_\pm(M_1) = i_\pm(A) + i_\pm(I_m - A) = i_\pm(I_m) + i_\pm\big(A - A^2\big),$$

$$i_\pm(M_2) = i_\pm(I_m + A) + i_\pm(I_m - A) = i_\pm(I_m) + i_\pm\big(I_m - A^2\big),$$

$$i_\pm(M_3) = i_\mp(I_m + A) + i_\pm(I_m - A) + i_\pm(A) = m + i_\pm\big(A - A^3\big).$$

Thus, we have (2.6)–(2.13). Applying Lemma 1.1 to (2.6), (2.7), (2.9), (2.10) and (2.12) yields (a)–(f). $\square$

Remark 2.2.

(a) Adding (2.6) and (2.7), (2.9) and (2.10), and the two equalities in (2.12) gives rise to the rank formulas in (2.1)–(2.3). In other words, the rank formulas in (2.1)–(2.3) can reasonably be separated into two groups of equalities for the partial inertias of $A - A^2$, $I_m - A^2$ and $A - A^3$. Namely, (2.6)–(2.13) are refinements of the rank formulas in (2.1)–(2.3) for a Hermitian matrix. Similarly, we can do such reasonable separations for many rank formulas for Hermitian matrix expressions; see [50] for more details.
(b) Notice that (2.6)–(2.13) are derived in about a paragraph by only using the well-known Sylvester's law of inertia and some trivial results in (1.7)–(1.9). Also note that the formulas in (2.6)–(2.13) and (2.1)–(2.3) are matched quite well. Hence, (2.6)–(2.13) and their variations should be proved/published in an earlier period of linear algebra, and thus become some classical contents on inertias of Hermitian matrices in linear algebra. As demonstrated in the following several sections, many simple and valuable results on ranks/inertias of Hermitian matrices can easily be proved by some elementary methods. This fact also shows that the theory of ranks/inertias of matrices was not so sufficiently developed in the past centuries that a huge amount of valuable problems that can be solved by the ranks and inertias of matrices were neglected.
(c) Theorem 2.1(a)–(f) demonstrates such a simple fact that once certain expansion formulas for the partial inertia of a Hermitian matrix expression are derived, we can use them to explicitly characterize the definiteness of the matrix expression. In Sections 3 and 4, we shall use Theorem 2.1(a)–(e) to obtain various conditional inequalities for orthogonal projectors and their operations in the Löwner partial ordering.
(d) The essential part of employing (2.15) is to find two different Hermitian matrices N_1 and N_2 that are congruent to M. Theoretically speaking, for any given matrices, we can use them to construct some block Hermitian matrices and to establish certain Hermitian congruence transformations for the block matrices, as demonstrated in (2.16)–(2.22). These Hermitian congruence transformations may or may not produce some informative reduced forms. Hence, they can or cannot be used to derive some acceptable results on algebraic properties of the given matrices. The three pairs of Hermitian congruence transformations in (2.17)–(2.22) are established according to the given matrix polynomials $A - A^2$, $I_m - A^2$ and $A - A^3$ and their three factors, and the main features of the right-hand sides of the six congruence transformations in (2.17)–(2.22) are block diagonal or skew diagonal. So that we are able to obtain the equalities for the inertias of $A - A^2$, $I_m - A^2$ and $A - A^3$ by using (1.8) and (1.9) to the six congruence transformations. This method was also successfully used in the author's recent paper [50].
(e) In the investigations of Hermitian matrices and applications, a popular method is using the spectral decompositions of the Hermitian matrices and their eigenvalues. However, this method is not so efficient when some matrices and their operations occur in the problems considered. The derivations of (2.6)–(2.13) show that algebraic properties of Hermitian matrices can also be derived without using the spectral decomposition of A and its eigenvalues.

Because (2.6)–(2.13) have such nice forms, it is worth trying to extend the expansion formulas to some general settings. For instance, it can be derived from (1.7) that the matrix pencil $\lambda A - \lambda^2 A^2$ satisfies

$$i_{\pm}\big(\lambda A - \lambda^2 A^2\big) = \begin{cases} i_{\pm}(A - \lambda A^2) & \text{if } \lambda > 0, \\ i_{\mp}(A - \lambda A^2) & \text{if } \lambda < 0 \end{cases} \tag{2.23}$$

for any Hermitian matrix A. Hence, we obtain from (2.6) and (2.7) the following result on the partial inertia of $A - \lambda A^2$ and the corresponding conditional matrix inequalities.

Corollary 2.3. *Let $A \in \mathbb{C}_{\mathrm{H}}^{m}$ and λ be a real number. Then,*

$$i_{+}\big(A - \lambda A^2\big) = i_{+}(I_m - \lambda A) + i_{+}(A) - m \quad \textit{for } \lambda > 0, \tag{2.24}$$

$$i_{-}\big(A - \lambda A^2\big) = i_{-}(I_m - \lambda A) + i_{-}(A) \quad \textit{for } \lambda > 0, \tag{2.25}$$

$$i_{+}\big(A - \lambda A^2\big) = i_{-}(I_m - \lambda A) + i_{+}(A) \quad \textit{for } \lambda < 0, \tag{2.26}$$

$$i_{-}\big(A - \lambda A^2\big) = i_{+}(I_m - \lambda A) + i_{-}(A) - m \quad \textit{for } \lambda < 0. \tag{2.27}$$

Hence, under $\lambda > 0$,

(a) $A - \lambda A^2 > 0$ $(A - \lambda A^2 \geqslant 0)$ *if and only if* $I_m > \lambda A > 0$ $(I_m \geqslant \lambda A \geqslant 0)$.
(b) $A - \lambda A^2 < 0$ $(A - \lambda A^2 \leqslant 0)$ *if and only if* $i_{-}(I_m - \lambda A) + i_{-}(A) = m$ $(i_{+}(I_m - \lambda A) + i_{+}(A) = m)$.

Under $\lambda < 0$,

(c) $A - \lambda A^2 > 0$ $(A - \lambda A^2 \geqslant 0)$ *if and only if* $i_{-}(I_m - \lambda A) + i_{+}(A) = m$ $(i_{+}(I_m - \lambda A) + i_{-}(A) = m)$.
(d) $A - \lambda A^2 < 0$ $(A - \lambda A^2 \leqslant 0)$ *if and only if* $I_m > \lambda A > 0$ $(I_m \geqslant \lambda A \geqslant 0)$.

Expansion formula for the partial inertia of a general quadratic matrix polynomial is given below.

Theorem 2.4. *Let $A \in \mathbb{C}_{\rm H}^{m}$, and let λ and μ be two real numbers with $\lambda\mu \neq 0$ and $\lambda < \mu$. Then,*

$$i_{+}\big[(\lambda I_m - A)(\mu I_m - A)\big] = i_{+}(\lambda I_m - A) + i_{-}(\mu I_m - A), \tag{2.28}$$

$$i_{-}\big[(\lambda I_m - A)(\mu I_m - A)\big] = i_{-}(\lambda I_m - A) + i_{+}(\mu I_m - A) - m. \tag{2.29}$$

Hence,

(a) $(\lambda I_m - A)(\mu I_m - A) > 0$ *if and only if* $i_{+}(\lambda I_m - A) + i_{-}(\mu I_m - A) = m$.
(b) $(\lambda I_m - A)(\mu I_m - A) \geqslant 0$ *if and only if* $i_{-}(\lambda I_m - A) + i_{+}(\mu I_m - A) = m$.
(c) $(\lambda I_m - A)(\mu I_m - A) < 0$ *if and only if* $\lambda I_m < A < \mu I_m$.
(d) $(\lambda I_m - A)(\mu I_m - A) \leqslant 0$ *if and only if* $\lambda I_m \leqslant A \leqslant \mu I_m$.

Proof. Let $B = \lambda I_m - A$ and $t = \mu - \lambda > 0$. Then, $(\lambda I_m - A)(\mu I_m - A)$ can be written as

$$(\lambda I_m - A)(\mu I_m - A) = B(tI_m + B) = tB + B^2. \tag{2.30}$$

Applying (2.24) and (2.25) to (2.30) gives

$$\begin{aligned} i_{+}\big(tB + B^2\big) &= i_{-}\big[(-B) - t^{-1}(-B)^2\big] = i_{-}\big(I_m + t^{-1}B\big) + i_{+}(B) \\ &= i_{+}(\lambda I_m - A) + i_{-}(\mu I_m - A), \\ i_{-}\big(tB + B^2\big) &= i_{+}\big[(-B) - t^{-1}(-B)^2\big] = i_{+}\big(I_m + t^{-1}B\big) + i_{-}(B) - m \\ &= i_{-}(\lambda I_m - A) + i_{+}(\mu I_m - A) - m, \end{aligned}$$

as required for (2.28) and (2.29). Results (a)–(d) follow from (2.28), (2.29) and Lemma 1.1. $\square$

Eqs. (2.28) and (2.29) can be combined as

$$i_{\pm}\big[(\lambda I_m - A)(\mu I_m - A)\big] = i_{\pm}(\lambda I_m - A) + i_{\mp}(\mu I_m - A) - i_{\pm}\big[(\lambda - \mu)I_m\big] \tag{2.31}$$

for any $\lambda\mu \neq 0$ and $\lambda < \mu$.

Also note that $A - A^3$ is a special case of the matrix polynomial

$$f(A) = (\lambda_1 I_m - A)(\lambda_2 I_m - A)(\lambda_3 I_m - A).$$

Hence, it is necessary to give some expansion formulas for the partial inertia of $f(A)$ in terms of the partial inertias of $\lambda_1 I_m - A$, $\lambda_2 I_m - A$ and $\lambda_3 I_m - A$ when A is Hermitian and $\lambda_1 < \lambda_2 < \lambda_3$. Further, recall that any real polynomial $f(x)$ has certain irreducible factorizations. Thus, it would be of interest to establish some expansion formulas for the partial inertia of a matrix polynomial $f(A)$ through its irreducible factorizations when A is Hermitian.

3. Expansion formulas for ranks/inertias of matrix pencils generated from two orthogonal projectors

Idempotent matrices and its special class—Hermitian idempotent matrices (orthogonal projectors) are considered as a simple but important class of matrices. In any case, idempotent matrices or orthogonal projectors and their algebraic aspects are interesting in themselves. Linear combinations of idempotent matrices or orthogonal projectors and their applications, as well as polynomials in idempotent matrices or orthogonal projectors and their algebraic properties were widely considered in the literature; see, e.g., [3–8,10,11,17,21,34,40,43,44,52,53,55,57,56,61]. It is well known from the spectral decomposition of a Hermitian matrix that any Hermitian matrix can be decomposed as a linear combination of certain mutually disjoint orthogonal projectors, or a linear combination of at most four orthogonal projections; see [39]. These facts prompt us to consider ranks/inertias of linear combinations of two or more orthogonal projectors. Just as the Hermitian congruence transformations given in (2.17)–(2.22), we are also able to construct some Hermitian congruence transformations for block matrices consisting of orthogonal projectors and their operations, and then use them to derive some expansion formulas for the rank/inertia of the linear combination of two orthogonal projectors of the same size.

For any two given matrices A and B of the same size and any two scalars a and b, the linear combination $aA + bB$ is often called a matrix pencil. The theory of matrix pencils is widely used in contemporary linear algebra and its applications; see, e.g., [2,15,19,20,31,33,46,47,58–60]. From the algebraic point of view, the mechanism of a general matrix pencil is not easy to distinguish. If, however, both P and Q are a pair of idempotent matrices or orthogonal projectors of the same size, then the matrix pencil $aP + bQ$ has many nice algebraic properties. By making use of certain elementary matrix operations for block matrices, it was shown in [55,57] that for any pair of idempotent matrices P and Q of the same size, and two scalars a and b such that $ab \neq 0$ and $a+b \neq 0$, the following expansion formulas for the ranks of $aP+bQ$, $P-Q$, $aPQ+bQP$ and $PQ-QP$ hold

$$r(aP+bQ) = r\begin{bmatrix} P & Q \\ Q & 0 \end{bmatrix} - r(Q), \tag{3.1}$$

$$r(P-Q) = r\begin{bmatrix} P \\ Q \end{bmatrix} + r[P, Q] - r(P) - r(Q), \tag{3.2}$$

$$r(aPQ+bQP) = r(P+Q) + r(PQ) + r(QP) - r(P) - r(Q), \tag{3.3}$$

$$r(PQ-QP) = r(P-Q) + r(I_m - P - Q) - m, \tag{3.4}$$

$$r(PQ-QP) = r\begin{bmatrix} P \\ Q \end{bmatrix} + r[P, Q] + r(PQ) + r(QP) - 2r(P) - 2r(Q). \tag{3.5}$$

These rank formulas were derived from some block matrix equalities consisting of P and Q. For instance, (3.4) can be derived from the following two decompositions

$$\begin{bmatrix} I_m & P+Q-I_m \\ P-Q & 0 \end{bmatrix} = \begin{bmatrix} I_m & I_m-2P \\ 0 & I_m \end{bmatrix}\begin{bmatrix} 0 & P+Q-I_m \\ P-Q & 0 \end{bmatrix}\begin{bmatrix} I_m & 0 \\ I_m-2Q & I_m \end{bmatrix},$$
$$\begin{bmatrix} I_m & P+Q-I_m \\ P-Q & 0 \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ P-Q & I_m \end{bmatrix}\begin{bmatrix} I_m & 0 \\ 0 & QP-PQ \end{bmatrix}\begin{bmatrix} I_m & P+Q-I_m \\ 0 & I_m \end{bmatrix}.$$

The rank expansion formulas in (3.1)–(3.5) can be used to characterize the nonsingularity of the matrix pencils $aP+bQ$ and $aPQ+bQP$, as well as to the equalities $P=Q$ and $PQ=QP$. In addition, for any pair of orthogonal projectors P and Q, the following rank equalities

$$r(PQ-QP) = 2r(PQ-PQP) = 2r(PQ-QPQ) = 2r\big[PQ-(PQ)^2\big] \tag{3.6}$$

hold; see [14]. Eq. (3.6) obviously implies that

$$PQ \text{ is an orthogonal projector} \quad\Leftrightarrow\quad (PQ)^2 = PQ \quad\Leftrightarrow\quad PQ = QP \quad\Leftrightarrow\quad PQ = PQP = QPQ. \tag{3.7}$$

In the investigation of orthogonal projectors, much attention has been paid to (simultaneous) decompositions of projectors and their operations. The well-known CS decomposition asserts that for a pair of orthogonal projectors P and Q of order m, there exists a unitary matrix U such that

$$P = U\operatorname{diag}\{I_{k_1}, 0_{k_2}, I_{k_3}, I_{k_4}, 0_{k_5}, 0_{k_6}\}U^*, \qquad Q = U\operatorname{diag}\left\{\begin{bmatrix} C^2 & CS \\ SC & S^2 \end{bmatrix}, I_{k_3}, 0_{k_4}, I_{k_5}, 0_{k_6}\right\}U^*, \tag{3.8}$$

where C and S are two positive diagonal matrices such that

$$C^2+S^2 = I_{k_1}, \qquad k_1+k_3+k_4 = r(P), \qquad r\begin{bmatrix} C^2 & CS \\ SC & S^2 \end{bmatrix} + k_3 + k_5 = r(Q), \qquad k_1+\cdots+k_6 = m;$$

see, e.g., [10,16,24]. Hence, the two products PQ and QP can be represented as

$$PQ = U\operatorname{diag}\left\{\begin{bmatrix} C^2 & CS \\ 0 & 0 \end{bmatrix}, I_{k_3}, 0\right\}U^*, \qquad QP = U\operatorname{diag}\left\{\begin{bmatrix} C^2 & 0 \\ SC & 0 \end{bmatrix}, I_{k_3}, 0\right\}U^*, \tag{3.9}$$

which was shown in [22]. Further,

$$P+Q = U\operatorname{diag}\left\{\begin{bmatrix} I_{k_1}+C^2 & CS \\ SC & S^2 \end{bmatrix}, 2I_{k_3}, I_{k_4}, I_{k_5}, 0_{k_6}\right\}U^*, \tag{3.10}$$

$$P-Q = U\operatorname{diag}\left\{\begin{bmatrix} S^2 & -CS \\ -SC & -S^2 \end{bmatrix}, 0_{k_3}, I_{k_4}, -I_{k_5}, 0_{k_6}\right\}U^*. \tag{3.11}$$

These decompositions can be used to obtain various algebraic properties for a pair of orthogonal projectors and their operations. The processes are, however, somewhat complicated in most situations. For three or more orthogonal projectors of the same size, it is hard to establish some simultaneous decompositions with informative structures. In this situation, we can only use the conventional operations for matrices to derive equalities for ranks/inertias of orthogonal projectors and their operations.

Note that the matrix pencil $aP+bQ$ is Hermitian if P and Q are both orthogonal projectors and a and b are both real. In this case, the rank/inertia of $aP+bQ$ may vary with respect to choice of the two real scalars a and b. Motivated by (3.1) and (3.2), we obtain the following results on the rank/inertia of $aP+bQ$ and their consequences.

Theorem 3.1. *Let $P, Q \in \mathbb{C}^m_{\mathrm{OP}}$, a and b be two real numbers with $ab \neq 0$ and $a+b \neq 0$. Then,*

$$i_\pm(aP+bQ) = i_\pm\begin{bmatrix} tP & Q \\ Q & 0 \end{bmatrix} + i_\mp(tP) - i_\mp(aP) - i_\mp(bQ), \tag{3.12}$$

$$i_\pm(aP+bQ) = i_\pm\begin{bmatrix} tQ & P \\ P & 0 \end{bmatrix} + i_\mp(tQ) - i_\mp(bQ) - i_\mp(aP), \tag{3.13}$$

$$r(aP+bQ) = r(P+Q) = r[P, Q], \tag{3.14}$$

where $t = a^{-1} + b^{-1}$. In particular,

(a) *$i_+(aP+bQ) = r[P, Q]$ and $i_-(aP+bQ) = 0$ if $a > 0$ and $b > 0$.*
(b) *$i_+(aP+bQ) = 0$ and $i_-(aP+bQ) = r[P, Q]$ if $a < 0$ and $b < 0$.*
(c) *$i_+(aP+bQ) = r(P)$ and $i_-(aP+bQ) = r[P, Q] - r(P)$ if $a > 0$, $b < 0$ and $a+b > 0$. In this case, $aP+bQ \leqslant 0$ if and only if $P = 0$; $aP+bQ \geqslant 0$ if and only if $\mathscr{R}(Q) \subseteq \mathscr{R}(P)$.*
(d) *$i_+(aP+bQ) = r[P, Q] - r(Q)$ and $i_-(aP+bQ) = r(Q)$ if $a > 0$, $b < 0$ and $a+b < 0$. In this case, $aP+bQ \leqslant 0$ if and only if $\mathscr{R}(P) \subseteq \mathscr{R}(Q)$; $aP+bQ \geqslant 0$ if and only if $Q = 0$.*
(e) *$i_+(aP+bQ) = r(Q)$ and $i_-(aP+bQ) = r[P, Q] - r(Q)$ if $a < 0$, $b > 0$ and $a+b > 0$. In this case, $aP+bQ \leqslant 0$ if and only if $Q = 0$; $aP+bQ \geqslant 0$ if and only if $\mathscr{R}(P) \subseteq \mathscr{R}(Q)$.*
(f) *$i_+(aP+bQ) = r[P, Q] - r(P)$ and $i_-(aP+bQ) = r(P)$ if $a < 0$, $b > 0$ and $a+b < 0$. In this case, $aP+bQ \leqslant 0$ if and only if $\mathscr{R}(Q) \subseteq \mathscr{R}(P)$; $aP+bQ \geqslant 0$ if and only if $P = 0$.*
(g) *The pencil $aP+bQ$ is nonsingular $\Leftrightarrow$ $P+Q$ is nonsingular $\Leftrightarrow$ $r[P, Q] = m$.*

Proof. Let

$$M = \begin{bmatrix} -aP & 0 & aP \\ 0 & -bQ & bQ \\ aP & bQ & 0 \end{bmatrix}, \qquad U = \begin{bmatrix} I_m & 0 & 0 \\ 0 & I_m & 0 \\ I_m & I_m & I_m \end{bmatrix}, \qquad V = \begin{bmatrix} I_m & 0 & -\frac{a}{2b}P \\ 0 & I_m & 2^{-1}P \\ \frac{b}{a+b}I_m & 0 & I_m - \frac{a}{2(a+b)}P \end{bmatrix}.$$

Then, M is obviously Hermitian. It is also easy to verify that both U and V are nonsingular, and

$$UMU^* = \begin{bmatrix} -aP & 0 & 0 \\ 0 & -bQ & 0 \\ 0 & 0 & aP+bQ \end{bmatrix}, \qquad VMV^* = \begin{bmatrix} -\frac{a}{b}(a+b)P & 0 & 0 \\ 0 & 0 & bQ \\ 0 & bQ & \frac{ab}{a+b}P \end{bmatrix}.$$

Applying (1.5)–(1.9) to the two equalities yields

$$i_\pm(M) = i_\pm(-aP) + i_\pm(-bQ) + i_\pm(aP+bQ) = i_\pm\left(-\frac{a}{b}(a+b)P\right) + i_\pm\begin{bmatrix} 0 & bQ \\ bQ & \frac{a}{b}(a+b)P \end{bmatrix},$$

that is,

$$i_\pm(aP+bQ) = i_\pm\begin{bmatrix} (a^{-1}+b^{-1})P & Q \\ Q & 0 \end{bmatrix} + i_\mp\big[(a^{-1}+b^{-1})P\big] - i_\mp(aP) - i_\mp(bQ),$$

where the two scalars $\frac{a}{b}(a+b)$ and $a^{-1}+b^{-1}$ have the same sign. Hence, (3.12) holds. Eq. (3.13) can be shown similarly. By (1.16) and (1.17),

$$i_+\begin{bmatrix} tP & Q \\ Q & 0 \end{bmatrix} = r[P, Q], \qquad i_-\begin{bmatrix} tQ & P \\ P & 0 \end{bmatrix} = r(P) \quad \text{for } t > 0, \tag{3.15}$$

$$i_+\begin{bmatrix} tP & Q \\ Q & 0 \end{bmatrix} = r(Q), \qquad i_-\begin{bmatrix} tQ & P \\ P & 0 \end{bmatrix} = r[P, Q] \quad \text{for } t < 0, \tag{3.16}$$

$$r\begin{bmatrix} tP & Q \\ Q & 0 \end{bmatrix} = r[P, Q] + r(Q) \quad \text{for } t \neq 0. \tag{3.17}$$

Adding the two equalities in (3.12) and applying (3.17) leads to (3.14). Applying Lemma 1.1, (1.7), (3.15) and (3.16) to (3.12) or (3.13) leads to (a)–(g). □

By recalling that $r(A) = \mathrm{tr}(A)$ if A is idempotent, we can replace the $r(P)$ and $r(Q)$ in Theorem 3.1 with $\mathrm{tr}(P)$ and $\mathrm{tr}(Q)$, respectively. Also, by recalling that $r(A) = r(B)$ and $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ if and only if $\mathscr{R}(A) = \mathscr{R}(B)$, we can rewrite the rank formula in (3.14) as

$$\mathscr{R}(aP+bQ) = \mathscr{R}[P, Q].$$

A special case of the pencil $aP+bQ$ is the difference $P-Q$, for which we have the following result (see also [50, Corollary 3.16]).

Theorem 3.2. *Let $P, Q \in \mathbb{C}^m_{\mathrm{OP}}$. Then,*

$$i_+(P-Q) = r[P, Q] - r(Q) = r(P - PQ), \tag{3.18}$$

$$i_-(P-Q) = r[P, Q] - r(P) = r(Q - PQ), \tag{3.19}$$

$$r(P-Q) = 2r[P, Q] - r(P) - r(Q) = r(P - PQ) + r(PQ - Q), \tag{3.20}$$

$$s(P-Q) = r(P) - r(Q). \tag{3.21}$$

Hence,

(a) $P - Q$ *is nonsingular if and only if* $r[P, Q] = r(P) + r(Q) = m$.
(b) $P = Q$ *if and only if* $\mathscr{R}(P) = \mathscr{R}(Q)$.
(c) $P > Q$ $(P < Q)$ *if and only if* $P = I_m$ *and* $Q = 0$ $(P = 0$ *and* $Q = I_m)$.
(d) $P \geqslant Q$ $(P \leqslant Q)$ *if and only if* $\mathscr{R}(Q) \subseteq \mathscr{R}(P)$ $(\mathscr{R}(P) \subseteq \mathscr{R}(Q))$.
(e) $r(P-Q) = r(P) + r(Q) \Leftrightarrow i_+(P-Q) = r(P) \Leftrightarrow i_-(P-Q) = r(Q) \Leftrightarrow \mathscr{R}(P) \cap \mathscr{R}(Q) = \{0\}$.
(f) $r(P-Q) = r(P) - r(Q) \Leftrightarrow i_+(P-Q) = r(P) - r(Q) \Leftrightarrow i_-(P-Q) = 0 \Leftrightarrow \mathscr{R}(Q) \subseteq \mathscr{R}(P)$.
(g) *The signature of* $P - Q$ *is zero if and only if* $r(P) = r(Q)$.

The results in Theorems 3.1 and 3.2 can also be given in some alternative forms. For instance, if P is an orthogonal projector of order m, then the difference $I_m - P$ is both Hermitian and idempotent, and thus it is an orthogonal projector as well and is often called the complementary orthogonal projector of P. Applying Theorem 3.2 to the difference $(I_m - P) - Q$ when both P and Q are orthogonal projector of order m, we obtain the following expansion formulas and their consequences.

Corollary 3.3. *Let $P, Q \in \mathbb{C}^m_{\mathrm{OP}}$. Then,*

$$i_+(I_m - P - Q) = m - r(P) - r(Q) + r(PQ), \tag{3.22}$$

$$i_-(I_m - P - Q) = r(PQ), \tag{3.23}$$

$$r(I_m - P - Q) = m - r(P) - r(Q) + 2r(PQ), \tag{3.24}$$

$$s(I_m - P - Q) = m - r(P) - r(Q). \tag{3.25}$$

Hence,

(a) $P + Q$ *has* t *eigenvalues equal to* 1, *where* $t = r(P) + r(Q) - 2r(PQ)$.
(b) $I_m - P - Q$ *is nonsingular if and only if* $r(PQ) = r(P) = r(Q)$.
(c) $P + Q = I_m \Leftrightarrow r(P) + r(Q) = m$ *and* $PQ = 0$.
(d) $I_m - P - Q > 0$ *if and only if* $P = Q = 0$.
(e) $I_m - P - Q < 0$ *if and only if* $P = Q = I_m$.
(f) $I_m - P - Q \geqslant 0$ *if and only if* $PQ = 0$.
(g) $I_m - P - Q \leqslant 0$ *if and only if* $r(PQ) = r(P) + r(Q) - m$.
(h) *The signature of* $I_m - P - Q$ *is zero if and only if* $r(P) + r(Q) = m$.

Proof. Applying (3.18) and (3.19) to the difference $(I_m - P) - Q$ yields

$$i_+(I_m - P - Q) = r[I_m - P, Q] - r(Q), \tag{3.26}$$

$$i_-(I_m - P - Q) = r[I_m - P, Q] - r(I_m - P) = r[I_m - P, Q] - m + r(P). \tag{3.27}$$

Applying (1.1) to $[I_m - P, Q]$ and simplifying, we obtain

$$r(I_m - P + Q) = r[I_m - P, Q] = r(I_m - P) + r\big[Q - (I_m - P)Q\big] = m - r(P) + r(PQ). \tag{3.28}$$

Inserting (3.28) into (3.26) and (3.27) produces (3.22) and (3.23). Eqs. (3.24) and (3.25) follow from (3.22) and (3.23). Results (a)–(h) follow from (3.22)–(3.25) and Lemma 1.1. □

For any two elements a and b in a ring, the two expressions $ab - ba$ and $ab + ba$ are often called the commutator and anti-commutator of a and b, respectively. The commutator and anti-commutator of two elements and their algebraic properties have been an attractive topic in noncommutative algebra. Note that $PQ + QP$ is Hermitian if both P and Q ate Hermitian, and that $PQ + QP$ can be written as $PQ + QP = (P + Q)^2 - (P + Q)$ if both P and Q are orthogonal projectors. Hence, we are able to derive from Theorems 2.1, 3.1 and Corollary 3.3 the following results.

Theorem 3.4. *Let $P, Q \in \mathbb{C}^m_{\mathrm{OP}}$. Then,*

$$i_+\big[(P+Q)^2-(P+Q)\big]=i_+(PQ+QP)=r(PQ), \tag{3.29}$$

$$i_-\big[(P+Q)^2-(P+Q)\big]=i_-(PQ+QP)=r[P,Q]-r(P)-r(Q)+r(PQ), \tag{3.30}$$

$$r\big[(P+Q)^2-(P+Q)\big]=r(PQ+QP)=r[P,Q]-r(P)-r(Q)+2r(PQ), \tag{3.31}$$

$$s\big[(P+Q)^2-(P+Q)\big]=s(PQ+QP)=r(P)+r(Q)-r[P,Q], \tag{3.32}$$

$$2i_-(PQ+QP)=r(PQ-QP). \tag{3.33}$$

Hence,

(a) *$(P+Q)^2-(P+Q)$ is nonsingular $\Leftrightarrow$ $PQ+QP$ is nonsingular $\Leftrightarrow$ $r[P,Q]=m$ and $r(PQ)=r(P)=r(Q)$.*
(b) *$(P+Q)^2>P+Q \Leftrightarrow PQ+QP>0 \Leftrightarrow P=Q=I_m$.*
(c) *$(P+Q)^2\geqslant P+Q \Leftrightarrow PQ+QP\geqslant 0 \Leftrightarrow PQ=QP \Leftrightarrow r[P,Q]=r(P)+r(Q)-r(PQ) \Leftrightarrow PQ\in\mathbb{C}^m_{\mathrm{OP}}$.*
(d) *$(P+Q)^2\leqslant P+Q \Leftrightarrow (P+Q)^2=P+Q \Leftrightarrow PQ+QP=0 \Leftrightarrow PQ=0$.*
(e) *The signature of $PQ+QP$ is zero if and only if $\mathscr{R}(P)\cap\mathscr{R}(Q)=\{0\}$.*

Proof. Note that $(P+Q)^2-(P+Q)=PQ+QP$. Consequently, applying (2.6) and (2.7) to this $(P+Q)^2-(P+Q)$ gives

$$i_+(PQ+QP)=i_+\big[(P+Q)^2-(P+Q)\big]=i_-(P+Q)+i_-(I_m-P-Q), \tag{3.34}$$

$$i_-(PQ+QP)=i_-\big[(P+Q)^2-(P+Q)\big]=i_+(P+Q)+i_+(I_m-P-Q)-m. \tag{3.35}$$

Substituting Lemma 3.1(a), (3.22) and (3.23) into (3.34) and (3.35), we obtain (3.29) and (3.30). Eqs. (3.31) and (3.32) follow from (3.29) and (3.30). Comparing (3.5) and (3.30) yields (3.33). Results (a)–(e) follow from (3.7), (3.29)–(3.33), and Lemma 1.1. □

The equivalence of $PQ+QP\geqslant 0$ and $PQ\in\mathbb{C}^m_{\mathrm{OP}}$ in Theorem 3.4(c) was given in [23]. Eqs. (3.29) and (3.30) are two expansion formulas for calculating the partial inertia of the anti-commutator of two orthogonal projectors. These two formulas show that the anti-commutator of two orthogonal projectors may have positive and negative eigenvalues simultaneously if both $PQ\neq 0$ and $r[P,Q]>r(P)+r(Q)-r(PQ)$. Thus, $PQ+QP$ is not definite in this case. A challenging task on the anti-commutator of two Hermitian matrices A and B is to give the distribution of the inertia triplet of $AB+BA$ under the conditions $A\geqslant 0$ and $B\geqslant 0$.

A generalization of (3.29) and (3.30) is given below.

Theorem 3.5. *Let $P, Q \in \mathbb{C}^m_{\mathrm{OP}}$. Then,*

$$i_\pm\big[(PQ)^k+(QP)^k\big]=i_\pm(PQ+QP) \tag{3.36}$$

for any integer $k\geqslant 2$.

Proof. It is easy to derive by induction that

$$(PQ)^k+(QP)^k=(P+Q)(P+Q-I_m)^{2k-1}=(P+Q-I_m)^{2k-1}(P+Q). \tag{3.37}$$

Applying (1.11) to (3.37) gives

$$\begin{aligned} i_\pm\big[(PQ)^k+(QP)^k\big]&=i_\pm\big[(P+Q)(P+Q-I_m)^{2k-1}\big]=i_\pm\big[(P+Q)(P+Q-I_m)\big]\\ &=i_\pm(PQ+QP), \end{aligned}$$

as required for (3.36). □

By a similar approach, we are also able to obtain the following expansion formulas for the partial inertias of some Hermitian polynomials in two orthogonal projectors.

Theorem 3.6. *Let $P, Q \in \mathbb{C}^m_{\mathrm{OP}}$. Then,*

$$i_+\big[(P+Q)^2-I_m\big]=r(PQ), \tag{3.38}$$

$$i_-\big[(P+Q)^2-I_m\big]=m-r(P)-r(Q)+r(PQ), \tag{3.39}$$

$$r\big[(P+Q)^2-I_m\big]=m-r(P)-r(Q)+2r(PQ), \tag{3.40}$$

$$s\big[(P+Q)^2-I_m\big]=r(P)+r(Q)-m. \tag{3.41}$$

Hence,

(a) $(P+Q)^2 - I_m$ *is nonsingular if and only if* $r(PQ) = r(P) = r(Q)$.
(b) $(P+Q)^2 - I_m > 0$ *if and only if* $P = Q = I_m$.
(c) $(P+Q)^2 - I_m \geqslant 0$ *if and only if* $r(PQ) = r(P) + r(Q) - m$.
(d) $(P+Q)^2 - I_m < 0$ *if and only if* $P = Q = 0$.
(e) $(P+Q)^2 - I_m \leqslant 0$ *if and only if* $PQ = 0$.
(f) $(P+Q)^2 = I_m$ *if and only if* $PQ = 0$ *and* $r(P) + r(Q) = m$.
(g) *The signature of* $(P+Q)^2 - I_m$ *is zero if and only if* $r(P) + r(Q) = m$.

Proof. Note that $(P+Q)^2 - I_m$ is Hermitian. Applying (2.9), (2.10), (3.22) and (3.23) to $(P+Q)^2 - I_m$, and simplifying by Theorem 3.1(a), (3.22) and (3.23), we obtain

$$i_+\big[(P+Q)^2 - I_m\big] = i_-(I_m + P + Q) + i_-(I_m - P - Q) = r(PQ),$$
$$i_-\big[(P+Q)^2 - I_m\big] = i_+(I_m + P + Q) + i_+(I_m - P - Q) - m = m - r(P) - r(Q) + r(PQ),$$

establishing (3.38) and (3.39). Eqs. (3.40) and (3.41) follow from (3.38) and (3.39). Results (a)–(g) follow from (3.38)–(3.41) and Lemma 1.1. □

Theorem 3.7. *Let* $P, Q \in \mathbb{C}^m_{\mathrm{OP}}$. *Then,*

$$i_+\big[(P+Q)^3 - (P+Q)\big] = r(PQ), \tag{3.42}$$
$$i_-\big[(P+Q)^3 - (P+Q)\big] = r[P, Q] - r(P) - r(Q) + r(PQ), \tag{3.43}$$
$$r\big[(P+Q)^3 - (P+Q)\big] = r[P, Q] - r(P) - r(Q) + 2r(PQ), \tag{3.44}$$
$$s\big[(P+Q)^3 - (P+Q)\big] = r(P) + r(Q) - r[P, Q], \tag{3.45}$$
$$2i_-\big[(P+Q)^3 - (P+Q)\big] = r(PQ - QP). \tag{3.46}$$

Hence,

(a) $(P+Q)^3 - (P+Q)$ *is nonsingular if and only if* $r[P, Q] = m$ *and* $r(PQ) = r(P) = r(Q)$.
(b) $(P+Q)^3 > (P+Q)$ *if and only if* $P = Q = I_m$.
(c) $(P+Q)^3 \geqslant P + Q \Leftrightarrow PQ = QP \Leftrightarrow r[P, Q] = r(P) + r(Q) - r(PQ) \Leftrightarrow PQ \in \mathbb{C}^m_{\mathrm{OP}}$.
(d) $(P+Q)^3 \leqslant P + Q \Leftrightarrow (P+Q)^3 = P + Q \Leftrightarrow PQ = 0$.
(e) *The signature of* $(P+Q)^3 - (P+Q)$ *is zero if and only if* $\mathscr{R}(P) \cap \mathscr{R}(Q) = \{0\}$.

Proof. Note that $(P+Q)^3 - (P+Q)$ is Hermitian. Applying (2.12) to this expression and simplifying by Theorem 3.1(a), (3.22) and (3.23), we obtain

$$i_+\big[(P+Q)^3 - (P+Q)\big] = i_-(P+Q) + i_+(I_m + P + Q) + i_-(I_m - P - Q) - m = r(PQ),$$
$$\begin{aligned} i_-\big[(P+Q)^3 - (P+Q)\big] &= i_+(P+Q) + i_-(I_m + P + Q) + i_+(I_m - P - Q) - m \\ &= r[P, Q] - r(P) - r(Q) + r(PQ), \end{aligned}$$

establishing (3.42) and (3.43). Eqs. (3.44) and (3.45) follow from (3.42) and (3.43). Comparing (3.5) and (3.43) yields (3.46). Results (a)–(e) follow from (3.7), (3.42)–(3.46) and Lemma 1.1. □

Observe that the right-hands of (3.29), (3.38) and (3.42) are the same, while the right-hands of (3.30) and (3.43) are the same. This fact allows us to conjecture that

$$i_+\big[(P+Q)^k - (P+Q)\big] = r(PQ), \qquad i_-\big[(P+Q)^k - (P+Q)\big] = r[P, Q] - r(P) - r(Q) + r(PQ)$$

for any integer $k \geqslant 2$.

Theorem 3.8. *Let* $P, Q \in \mathbb{C}^m_{\mathrm{OP}}$. *Then,*

$$i_+\big[(P-Q)^2 - (P-Q)\big] = r[P, Q] - r(P), \tag{3.47}$$
$$i_-\big[(P-Q)^2 - (P-Q)\big] = r[P, Q] - r(P) - r(Q) + r(PQ), \tag{3.48}$$
$$r\big[(P-Q)^2 - (P-Q)\big] = 2r[P, Q] - 2r(P) - r(Q) + r(PQ), \tag{3.49}$$

$$s\big[(P-Q)^2-(P-Q)\big]=r(Q)-r(PQ), \tag{3.50}$$

$$2i_-\big[(P-Q)^2-(P-Q)\big]=r(PQ-QP). \tag{3.51}$$

Hence,

(a) $(P-Q)^2-(P-Q)$ *is nonsingular if and only if* $2r[P,Q]=2r(P)+r(Q)-r(PQ)+m$.
(b) $(P-Q)^2-(P-Q)>0$ *if and only if* $P=0$ *and* $Q=I_m$.
(c) $(P-Q)^2-(P-Q)\geqslant 0 \Leftrightarrow PQ=QP \Leftrightarrow r[P,Q]=r(P)+r(Q)-r(PQ) \Leftrightarrow PQ\in\mathbb{C}^m_{\mathrm{OP}}$.
(d) $(P-Q)^2-(P-Q)\leqslant 0 \Leftrightarrow (P-Q)^2=(P-Q) \Leftrightarrow PQ=Q \Leftrightarrow \mathscr{R}(Q)\subseteq\mathscr{R}(P)$.
(e) *The signature of* $(P-Q)^2-(P-Q)$ *is zero if and only if* $r(Q)=r(PQ)$.

Proof. Note that $(P-Q)^2-(P-Q)$ is Hermitian. Applying (2.6) and (2.7) to $(P-Q)^2-(P-Q)$ and simplifying by (3.18), (3.19) and (3.28), we obtain

$$\begin{aligned}
i_+\big[(P-Q)^2-(P-Q)\big]&=i_-(P-Q)+i_-(I_m-P+Q)=r[P,Q]-r(P),\\
i_-\big[(P-Q)^2-(P-Q)\big]&=i_+(P-Q)+i_+(I_m-P+Q)-m\\
&=r[P,Q]-r(Q)+m-r(P)+r(PQ)-m\\
&=r[P,Q]-r(Q)-r(P)+r(PQ),
\end{aligned}$$

establishing (3.47) and (3.48). Eqs. (3.49) and (3.50) follow from (3.47) and (3.48). Comparing (3.5) and (3.48) yields (3.51). Results (a)–(e) follow from (3.7), (3.47)–(3.51) and Lemma 1.1. □

Theorem 3.9. *Let* $P,Q\in\mathbb{C}^m_{\mathrm{OP}}$. *Then,*

$$i_-\big[(P-Q)^2-I_m\big]=r\big[(P-Q)^2-I_m\big]=m-r(P)-r(Q)+2r(PQ). \tag{3.52}$$

Hence,

(a) $(P-Q)^2-I_m$ *is nonsingular if and only if* $r(PQ)=r(P)=r(Q)$.
(b) $(P-Q)^2\leqslant I_m$ *always holds.*
(c) $(P-Q)^2\geqslant I_m \Leftrightarrow (P-Q)^2=I_m \Leftrightarrow PQ=0$ *and* $r(P)+r(Q)=m$.
(d) $(P-Q)^2$ *has* t *eigenvalues equal to* 1, *where* $t=r(P)+r(Q)-2r(PQ)$.

Proof. Note that $(P-Q)^2-I_m$ is Hermitian. Applying (2.9) and (2.10) to $(P-Q)^2-I_m$ and simplifying by (3.18), (3.19) and (3.28), we obtain

$$\begin{aligned}
i_+\big[(P-Q)^2-I_m\big]&=i_-(I_m+P-Q)+i_-(I_m-P+Q)=0,\\
i_-\big[(P-Q)^2-I_m\big]&=i_+(I_m+P-Q)+i_+(I_m-P+Q)-m\\
&=r[P,I_m-Q]+r[I_m-P,Q]-m=m-r(Q)-r(P)+2r(PQ),
\end{aligned}$$

establishing (3.52). Results (a)–(c) follow from (3.52) and Lemma 1.1. □

Theorem 3.10. *Let* $P,Q\in\mathbb{C}^m_{\mathrm{OP}}$. *Then,*

$$i_\pm\big[(P-Q)^3-(P-Q)\big]=i_\pm(QPQ-PQP)=r[P,Q]-r(P)-r(Q)+r(PQ), \tag{3.53}$$

$$r\big[(P-Q)^3-(P-Q)\big]=r(QPQ-PQP)=2r[P,Q]-2r(P)-2r(Q)+2r(PQ), \tag{3.54}$$

$$s\big[(P-Q)^3-(P-Q)\big]=s(QPQ-PQP)=0, \tag{3.55}$$

$$r(QPQ-PQP)=r(PQ-QP). \tag{3.56}$$

Hence,

$$\begin{aligned}
(P-Q)^3\geqslant P-Q \quad&\Leftrightarrow\quad (P-Q)^3\leqslant P-Q \quad\Leftrightarrow\quad (P-Q)^3=P-Q \quad\Leftrightarrow\quad QPQ=PQP\\
&\Leftrightarrow\quad PQ=QP \quad\Leftrightarrow\quad r[P,Q]=r(P)+r(Q)-r(PQ) \quad\Leftrightarrow\quad PQ\in\mathbb{C}^m_{\mathrm{OP}}.
\end{aligned} \tag{3.57}$$

Proof. Note that $(P-Q)^3-(P-Q)$ is Hermitian and it is easily verified that $(P-Q)^3-(P-Q)=QPQ-PQP$. Applying (2.12) to this equality and simplifying by (3.18), (3.19) and (3.28), we obtain

$$\begin{aligned} i_+\big[(P-Q)^3-(P-Q)\big] &= i_+(I_m+P-Q)+i_-(I_m-P+Q)+i_-(P-Q)-m \\ &= r[P,Q]-r(P)-r(Q)+r(PQ), \\ i_-\big[(P-Q)^3-(P-Q)\big] &= i_-(I_m+P-Q)+i_+(I_m-P+Q)+i_+(P-Q)-m \\ &= r[P,Q]-r(Q)-r(P)+r(PQ), \end{aligned}$$

establishing (3.53). Eqs. (3.54) and (3.55) follow from (3.53). Comparing (3.5) and (3.54) yields (3.56). Setting the right-hands of (3.53), (3.54) and (3.56) to zero leads to the equivalence in (3.57). □

Theorem 3.11. *Let $P, Q \in \mathbb{C}^m_{\mathrm{OP}}$. Then,*

$$i_+(QPQ+PQP)=r(QPQ+PQP)=r(PQ). \tag{3.58}$$

Proof. It is easy to verify that

$$PQP+QPQ=(P+Q)(P+Q-I_m)^2=(P+Q-I_m)^2(P+Q). \tag{3.59}$$

Also note that $P+Q \geqslant 0$. Then, we find by (1.11) and (3.29) that

$$\begin{aligned} i_+(PQP+QPQ) &= r(PQP+QPQ)=i_+\big[(P+Q)(P+Q-I_m)^2\big] \\ &= i_+\big[(P+Q)(P+Q-I_m)\big]=i_+(PQ+QP)=r(PQ), \end{aligned}$$

as required for (3.58). □

Further, we have the following results on $P \pm PQP$ and $I_m - PQP$.

Theorem 3.12. *Let $P, Q \in \mathbb{C}^m_{\mathrm{OP}}$. Then,*

$$i_+(P+PQP)=r(P+PQP)=r(P), \tag{3.60}$$

$$i_+(P-PQP)=r(P-PQP)=r[P,Q]-r(Q), \tag{3.61}$$

$$i_+(Q+PQP)=r(Q+PQP)=r[P,Q]-r(P)+r(PQ), \tag{3.62}$$

$$i_+(Q-PQP)=r[P,Q]-r(P), \tag{3.63}$$

$$i_-(Q-PQP)=r[P,Q]-r(P)-r(Q)+r(PQ), \tag{3.64}$$

$$i_+(I_m-PQP)=r(I_m-PQP)=r[P,Q]-r(P)-r(Q)+m. \tag{3.65}$$

Hence,

(a) $P \leqslant PQP \Leftrightarrow P = PQP \Leftrightarrow \mathscr{R}(P) \subseteq \mathscr{R}(Q)$.
(b) $Q \leqslant PQP \Leftrightarrow Q = PQP \Leftrightarrow \mathscr{R}(Q) \subseteq \mathscr{R}(P)$.
(c) $Q \geqslant PQP \Leftrightarrow r[P,Q]=r(P)+r(Q)-r(PQ)$.
(d) *PQP has t eigenvalues equal to 1, where $t=r(P)+r(Q)-r[P,Q]$.*
(e) *$I_m \geqslant PQP$ always holds, and $I_m > PQP \Leftrightarrow \mathscr{R}(P)\cap\mathscr{R}(Q)=\{0\}$.*

Proof. Note that $I_m+Q>0$ and $I_m-Q\geqslant 0$. Therefore, $P(I_m \pm Q)P = P \pm PQP \geqslant 0$. Consequently, we obtain from (1.15) that

$$\begin{aligned} i_+(P+PQP) &= r(P+PQP)=r(P), \\ i_+(P-PQP) &= r(P-PQP)=r\begin{bmatrix} P & PQ \\ QP & Q \end{bmatrix}-r(Q) \\ &= r\big([P,Q]^*[P,Q]\big)-r(Q)=r[P,Q]-r(Q), \end{aligned}$$

as required for (3.60) and (3.61). Note that $Q+PQP \geqslant 0$. Therefore, applying (1.2), (1.4) and (3.14), and simplifying by EBMOs, we obtain

$$\begin{aligned} i_+(Q+PQP) &= r(Q+PQP)=r[Q, PQP]=r[Q, PQ]=r[Q-PQ, PQ] \\ &= r(Q-PQ)+r(PQ)=r[P,Q]-r(P)+r(PQ) \quad \big(\text{by (1.2)}\big), \end{aligned}$$

as required for (3.62). Applying (1.14) and $Q-QPQPQ \geqslant 0$, and simplifying by EBMOs, we obtain

$$
\begin{aligned}
i_+(Q-PQP) &= i_+\begin{bmatrix} Q & PQ \\ QP & Q \end{bmatrix} - i_+(Q) = i_+\begin{bmatrix} 0 & PQ-QPQ \\ QP-QPQ & Q-QPQPQ \end{bmatrix} \\
&= r[QP-QPQ,\ Q-QPQPQ] \quad \big(\text{by (1.16)}\big) \\
&= r[QP-QPQ,\ Q-QPQ] = r[Q-QP,\ Q-QPQ] \\
&= r[Q-QP, 0] = r[P, Q] - r(P) \quad \big(\text{by (1.2)}\big),
\end{aligned}
$$

$$
\begin{aligned}
i_-(Q-PQP) &= i_-\begin{bmatrix} Q & PQ \\ QP & Q \end{bmatrix} - i_-(Q) = i_-\begin{bmatrix} 0 & PQ-QPQ \\ QP-QPQ & Q-QPQPQ \end{bmatrix} \\
&= r(PQ-QPQ) \quad \big(\text{by (1.16)}\big) \\
&= r[Q, PQ] - r(Q) \quad \big(\text{by (1.2)}\big) \\
&= r[Q-PQ, PQ] - r(Q) = r(Q-PQ) + r(PQ) - r(Q) \\
&= r[P, Q] - r(P) - r(Q) + r(PQ) \quad \big(\text{by (1.2)}\big),
\end{aligned}
$$

as required for (3.63) and (3.64). Applying (1.18) to $I_m - PQP$ and simplifying by (3.61), we obtain

$$
\begin{aligned}
i_+(I_m-PQP) &= i_+\begin{bmatrix} Q & QP \\ PQ & I_m \end{bmatrix} - i_+(Q) = i_+(Q-QPQ) + m - r(Q) \\
&= r[P, Q] - r(P) - r(Q) + m, \\
i_-(I_m-PQP) &= i_-\begin{bmatrix} Q & QP \\ PQ & I_m \end{bmatrix} - i_-(Q) = i_-(Q-QPQ) + I_-(I_m) = 0,
\end{aligned}
$$

establishing (3.65). □

Hermitian matrix polynomials generated from two orthogonal projectors can be formulated arbitrarily. It seems from the previous results that some expansion formulas for the inertias of these Hermitian matrix polynomials can always be derived with some effort.

The matrix product PQ, the matrix pencil $aPQ + bQP$ and the commutator $PQ - QP$ are not necessarily Hermitian even both P and Q are orthogonal projectors. Hence, the rank formulas in (3.3)–(3.6) cannot be refined to the situations for inertia. However, if both P and Q are orthogonal projectors, the complex matrix $j(PQ - QP)$ is Hermitian, where $j = \sqrt{-1}$. It can be seen from (3.5) that a reasonable conjecture on the partial inertia of $j(PQ - QP)$ is given by

$$i_\pm\big[j(PQ-QP)\big] = r[P, Q] - r(P) - r(Q) + r(PQ).$$

4. Expansion formulas for ranks/inertias of matrix pencils generated from three or more orthogonal projectors

Matrix pencils can be generated from linear combinations of three or more matrices. In fact, the matrix expression $I_m - P - Q$ in Corollary 3.3 is a special case of the matrix pencils generated from the three orthogonal projectors I_m, P and Q. The mechanism of these pencils, however, are quite complicated in general. In what follows, we give some expansion formulas for the rank/inertia of a matrix pencil generated from three orthogonal projectors under some conditions.

Theorem 4.1. *Assume that $P, P_1, P_2 \in \mathbb{C}^m_{\mathrm{OP}}$ satisfy*

$$\mathscr{R}(P_1) \subseteq \mathscr{R}(P) \quad \text{and} \quad \mathscr{R}(P_2) \subseteq \mathscr{R}(P), \tag{4.1}$$

and λ, λ_1 and λ_2 are nonzero real numbers. Also, denote

$$D = \mathrm{diag}\{P_1, P_2\}, \qquad G = \begin{bmatrix} t_1 I_m & \lambda^{-1} I_m \\ \lambda^{-1} I_m & t_2 I_m \end{bmatrix}, \tag{4.2}$$

where $t_1 = \lambda_1^{-1} + \lambda^{-1}$ and $t_2 = \lambda_2^{-1} + \lambda^{-1}$. Then,

$$i_\pm(\lambda P + \lambda_1 P_1 + \lambda_2 P_2) = i_\pm(\lambda P) - i_\mp(\lambda_1 P_1) - i_\mp(\lambda_2 P_2) + i_\mp(DGD), \tag{4.3}$$

$$r(\lambda P + \lambda_1 P_1 + \lambda_2 P_2) = r(P) - r(P_1) - r(P_2) + r(DGD). \tag{4.4}$$

Hence,

(a) *If* $G > 0$, *i.e.*, $\lambda\lambda_1(\lambda+\lambda_1) > 0$, $\lambda\lambda_2(\lambda+\lambda_2) > 0$ *and* $\lambda\lambda_1\lambda_2(\lambda+\lambda_1+\lambda_2) > 0$, *then*

$$i_+(\lambda P + \lambda_1 P_1 + \lambda_2 P_2) = i_+(\lambda P) - i_-(\lambda_1 P_1) - i_-(\lambda_2 P_2), \tag{4.5}$$

$$i_-(\lambda P + \lambda_1 P_1 + \lambda_2 P_2) = i_-(\lambda P) - i_+(\lambda_1 P_1) - i_+(\lambda_2 P_2) + r(P_1) + r(P_2), \tag{4.6}$$

$$r(\lambda P + \lambda_1 P_1 + \lambda_2 P_2) = r(P). \tag{4.7}$$

(b) *If* $G < 0$, *i.e.*, $\lambda\lambda_1(\lambda+\lambda_1) < 0$, $\lambda\lambda_2(\lambda+\lambda_2) < 0$ *and* $\lambda\lambda_1\lambda_2(\lambda+\lambda_1+\lambda_2) > 0$, *then*

$$i_+(\lambda P + \lambda_1 P_1 + \lambda_2 P_2) = i_+(\lambda P) - i_-(\lambda_1 P_1) - i_-(\lambda_2 P_2) + r(P_1) + r(P_2), \tag{4.8}$$

$$i_-(\lambda P + \lambda_1 P_1 + \lambda_2 P_2) = i_-(\lambda P) - i_+(\lambda_1 P_1) - i_+(\lambda_2 P_2), \tag{4.9}$$

$$r(\lambda P + \lambda_1 P_1 + \lambda_2 P_2) = r(P). \tag{4.10}$$

(c) $P - P_1 - P_2$ *satisfies the following equalities*

$$i_+(P - P_1 - P_2) = r(P) - r(P_1) - r(P_2) + r(P_1P_2), \tag{4.11}$$

$$i_-(P - P_1 - P_2) = r(P_1P_2), \tag{4.12}$$

$$r(P - P_1 - P_2) = r(P) - r(P_1) - r(P_2) + 2r(P_1P_2), \tag{4.13}$$

$$s(P - P_1 - P_2) = r(P) - r(P_1) - r(P_2). \tag{4.14}$$

Hence,

(i) $P - P_1 - P_2$ *is nonsingular if and only if* $r(P) = r(P_1) + r(P_2) - 2r(P_1P_2) + m$.
(ii) $P - P_1 - P_2 \geqslant 0$ *if and only if* $P_1P_2 = 0$.
(iii) $P - P_1 - P_2 \leqslant 0$ *if and only if* $r(P) = r(P_1) + r(P_2) - r(P_1P_2)$.
(iv) $P = P_1 + P_2$ *if and only if* $P_1P_2 = 0$ *and* $r(P) = r(P_1) + r(P_2)$.
(v) *The signature of* $P - P_1 - P_2$ *is zero if and only if* $r(P) = r(P_1) + r(P_2)$.

(d) $2P - P_1 - P_2$ *satisfies the following equalities*

$$i_+(2P - P_1 - P_2) = r(2P - P_1 - P_2) = r(P) - r(P_1) - r(P_2) + r[P_1, P_2]. \tag{4.15}$$

Hence, $2P = P_1 + P_2$ *if and only if* $r(P) = r(P_1) + r(P_2) - r[P_1, P_2]$.

Proof. Let

$$M = \begin{bmatrix} -\lambda^{-1}P & 0 & 0 & P \\ 0 & -\lambda_1^{-1}P_1 & 0 & P_1 \\ 0 & 0 & -\lambda_2^{-1}P_2 & P_2 \\ P & P_1 & P_2 & 0 \end{bmatrix}, \tag{4.16}$$

$$U = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & I_m & 0 \\ \lambda I_m & \lambda_1 I_m & \lambda_2 I_m & I_m \end{bmatrix}, \qquad V = \begin{bmatrix} I_m & 0 & 0 & \frac{\lambda^{-1}}{2}P \\ -P_1 & I_m & 0 & -\lambda^{-1}P_1 \\ -P_2 & 0 & I_m & -\lambda^{-1}P_2 \\ 0 & 0 & 0 & I_m \end{bmatrix}. \tag{4.17}$$

Then, it is easily verified that U and V are two nonsingular matrices, and from $PP_1 = P_1P = P_1$ and $PP_2 = P_2P = P_2$ that

$$UMU^* = \begin{bmatrix} -\lambda^{-1}P & 0 & 0 & 0 \\ 0 & -\lambda_1^{-1}P_1 & 0 & 0 \\ 0 & 0 & -\lambda_2^{-1}P_2 & 0 \\ 0 & 0 & 0 & \lambda P + \lambda_1 P_1 + \lambda_1 P_2 \end{bmatrix}, \tag{4.18}$$

$$VMV^* = \begin{bmatrix} 0 & 0 & 0 & P \\ 0 & -(\lambda_1^{-1} + \lambda^{-1})P_1 & -\lambda^{-1}P_1P_2 & 0 \\ 0 & -\lambda^{-1}P_2P_1 & -(\lambda_2^{-1} + \lambda^{-1})P_2 & 0 \\ P & 0 & 0 & 0 \end{bmatrix}. \tag{4.19}$$

Applying (1.5)–(1.9) to (4.18) and (4.19) yields

$$\begin{aligned} i_\pm(M) &= i_\mp(\lambda^{-1}P) + i_\mp(\lambda_1^{-1}P_1) + i_\mp(\lambda_2^{-1}P_2) + i_\pm(\lambda P + \lambda_1 P_1 + \lambda_2 P_2) \\ &= r(P) + i_\mp \begin{bmatrix} t_1 P_1 & \lambda^{-1}P_1P_2 \\ \lambda^{-1}P_2P_1 & t_2P_2 \end{bmatrix} = r(P) + i_\mp(DGD), \end{aligned}$$

establishing (4.3) and (4.4).

If $G > 0$, then $DGD \geqslant 0$ and $i_+(DGD) = r(DGD) = r(D) = r(P_1) + r(P_2)$; if $G < 0$, then $DGD \leqslant 0$ and $i_-(DGD) = r(DGD) = r(D) = r(P_1) + r(P_2)$. Hence, (4.3) and (4.4) reduce to (4.5)–(4.7) and (4.8)–(4.10), respectively.

Let $\lambda = -\lambda_1 = -\lambda_2 = 1$. Then, it follows from (1.9) that

$$i_\mp(DGD) = i_\mp\begin{bmatrix} t_1P_1 & \lambda^{-1}P_1P_2 \\ \lambda^{-1}P_2P_1 & t_2P_2 \end{bmatrix} = i_\pm\begin{bmatrix} 0 & P_1P_2 \\ P_2P_1 & 0 \end{bmatrix} = r(P_1P_2),$$

so that (4.3) reduces to (4.11) and (4.12). Eqs. (4.13) and (4.14) follow from (4.3) and (4.4). Results (i)–(iv) in (c) are direct consequences of (4.3)–(4.14). Let $2^{-1}\lambda = -\lambda_1 = -\lambda_2 = 1$. Then,

$$i_\pm(DGD) = i_\pm\begin{bmatrix} t_1P_1 & \lambda^{-1}P_1P_2 \\ \lambda^{-1}P_2P_1 & t_2P_2 \end{bmatrix} = i_\pm\begin{bmatrix} 2P_1 & 2P_1P_2 \\ 2P_2P_1 & 2P_2 \end{bmatrix} = i_\pm\big([P_1, P_2]^*[P_1, P_2]\big),$$

so that (4.3) reduces to (4.15). □

The two range inclusions in (4.1) are a reasonable assumption on the relations among three orthogonal projectors, for instance, the orthogonal projectors onto the ranges of $M = [A, B]$ and its two submatrices A and B satisfy such conditions. The results in Theorem 4.1, as well as the constructions of (4.16)–(4.19) show that to obtain satisfactory expansion formulas for the inertia of a general matrix pencil generated from three or more orthogonal projectors is a challenging task. Setting $P = I_m$ in (4.3) and (4.4), we obtain the following result.

Corollary 4.2. *Let $P, Q \in \mathbb{C}^m_{\mathrm{OP}}$, λ, λ_1 and λ_2 be nonzero real numbers, and denote*

$$M = \begin{bmatrix} t_1P & \lambda^{-1}PQ \\ \lambda^{-1}QP & t_2Q \end{bmatrix}, \quad t_1 = \lambda_1^{-1} + \lambda^{-1} \text{ and } t_2 = \lambda_2^{-1} + \lambda^{-1}.$$

Then,

$$i_\pm(\lambda I_m + \lambda_1P + \lambda_2Q) = i_\pm(\lambda I_m) - i_\mp(\lambda_1P) - i_\mp(\lambda_2Q) + i_\mp(M), \tag{4.20}$$

$$r(\lambda I_m + \lambda_1P + \lambda_2Q) = m - r(P) - r(Q) + r(M). \tag{4.21}$$

Hence,

(a) *The linear matrix inequality $\lambda I_m + \lambda_1P + \lambda_2Q > 0$ (< 0) holds if and only if*

$$i_-(M) = m - i_+(\lambda I_m) + i_-(\lambda_1P) + i_-(\lambda_2Q) \quad \big(i_+(M) = m - i_-(\lambda I_m) + i_+(\lambda_1P) + i_+(\lambda_2Q)\big).$$

(b) *The linear matrix inequality $\lambda I_m + \lambda_1P + \lambda_2Q \geqslant 0$ ($\leqslant 0$) holds if and only if*

$$i_+(M) = i_+(\lambda_1P) + i_+(\lambda_2Q) - i_-(\lambda I_m) \quad \big(i_-(M) = i_-(\lambda_1P) + i_-(\lambda_2Q) - i_+(\lambda I_m)\big).$$

(c) *$\lambda I_m + \lambda_1P + \lambda_2Q = 0$ if and only if $r(M) = r(P) + r(Q) - m$.*

(d) *$r(\lambda I_m + \lambda_1P + \lambda_2Q) < m$, i.e., $|\lambda I_m + \lambda_1P + \lambda_2Q| = 0$ if and only if $r(M) < r(P) + r(Q)$.*

Applying Theorem 2.1 and Corollary 4.2 to some matrix polynomials of the matrix pencil $\lambda I_m + \lambda_1P + \lambda_2Q$, such as,

$$(\lambda I_m + \lambda_1P + \lambda_2Q) - (\lambda I_m + \lambda_1P + \lambda_2Q)^2, \qquad I_m - (\lambda I_m + \lambda_1P + \lambda_2Q)^2,$$
$$(\lambda I_m + \lambda_1P + \lambda_2Q) - (\lambda I_m + \lambda_1P + \lambda_2Q)^3$$

will yield a variety of expansion formulas for the partial inertias of the matrix expressions. We leave this work for the reader.

When considering a pair of orthogonal projectors $P, Q \in \mathbb{C}^m_{\mathrm{OP}}$, it is usually assumed that P and Q satisfy certain equalities, such as, $PQ = QP$, $PQP = P$, $PQP = Q$, $PQP = QPQ$, etc. In these cases, all the linear combinations of the two orthogonal projectors and their possible products generate a finite-dimensional (non-)commutative algebra over the real or complex field. For instance, if the pair of orthogonal projectors satisfy $PQ = QP$, then the corresponding matrix pencil

$$M = a_0I_m + a_1P + a_2Q + a_3PQ, \quad a_0, a_1, a_2, a_3 \in \mathbb{R} \tag{4.22}$$

is Hermitian, and all these matrix pencils generate a commutative algebra up to four dimensions over the real number field $\mathbb{R}$ under the conventional addition and multiplication of matrices. This matrix algebra has many interesting properties. A remarkable universal similarity factorization equality (USFE) associated with the pencil is given by

$$L\,\mathrm{diag}(M_1, M_2, M_3, M_4)L^{-1} = \mathrm{diag}(t_1I_m, t_2I_m, t_3I_m, t_4I_m), \tag{4.23}$$

where

$$M_1 = a_0 I_m + a_1 P + a_2 Q + a_3 PQ,$$
$$M_2 = (a_0 + a_2) I_m + (a_1 + a_3) P - a_2 Q - a_3 PQ,$$
$$M_3 = (a_0 + a_1) I_m - a_1 P + (a_2 + a_3) Q - a_3 PQ,$$
$$M_4 = (a_0 + a_1 + a_2 + a_3) I_m - (a_1 + a_3) P - (a_2 + a_3) Q + a_3 PQ,$$
$$t_1 = a_0, \qquad t_2 = a_0 + a_2, \qquad t_3 = a_0 + a_1, \qquad t_4 = a_0 + a_1 + a_2 + a_3,$$

and

$$L = L^* = L^{-1} = \begin{bmatrix} L_1 & L_2 & L_3 & L_4 \\ L_2 & L_1 & -L_4 & -L_3 \\ L_3 & -L_4 & L_1 & -L_2 \\ L_4 & -L_3 & -L_2 & L_1 \end{bmatrix},$$

in which the four matrices

$$L_1 = I_m - P - Q + PQ, \qquad L_2 = Q - PQ, \qquad L_3 = P - PQ, \qquad L_4 = PQ \tag{4.24}$$

satisfy

$$L_1 + L_2 + L_3 + L_4 = I_m, \qquad L_i^2 = L_i = L_i^*, \qquad L_i L_j = 0, \quad i \neq j,\ i, j = 1, \ldots, 4, \tag{4.25}$$
$$r(L_1) = m - r(P) - r(Q) + r(PQ), \qquad r(L_2) = r(Q) - r(PQ), \qquad r(L_3) = r(P) - r(PQ); \tag{4.26}$$

see [53,57,56]. It can be derived from (4.23) that the matrix pencil in (4.22) can be decomposed as the following linear combination of the four orthogonal projectors $L_1, \ldots, L_4$:

$$M = t_1 L_1 + t_2 L_2 + t_3 L_3 + t_4 L_4 = \widehat{L}\,\mathrm{diag}(t_1 I_m, t_2 I_m, t_3 I_m, t_4 I_m)\widehat{L}^*, \tag{4.27}$$

where the row block matrix $L = [L_1, L_2, L_3, L_4]$ satisfies $LL^* = I_m$. Notice the matrices $L_1, \ldots, L_4$ in (4.24) are four mutually disjoint orthogonal projectors, and the four scalars $t_1, \ldots, t_4$ are the eigenvalues of M. Therefore, (4.27) is in fact a closed-form spectral decomposition of the matrix pencil (4.22), which can also be called a disjoint orthogonal projection decomposition (DOPD) of the matrix pencil M in (4.22). Many consequences can be derived from the DOPD in (4.27). For instance,

(i) The expansion formulas for the partial inertia of the matrix pencil M in (4.22) are

$$\begin{aligned} i_{\pm}(M) &= i_{\pm}(t_1) r(L_1) + i_{\pm}(t_2) r(L_1) + i_{\pm}(t_3) r(L_3) + i_{\pm}(t_4) r(L_4) \\ &= i_{\pm}(t_1) m + \big[i_{\pm}(t_3) - i_{\pm}(t_1)\big] r(P) + \big[i_{\pm}(t_2) - i_{\pm}(t_1)\big] r(Q) \\ &\quad + \big[i_{\pm}(t_1) - i_{\pm}(t_2) - i_{\pm}(t_3) + i_{\pm}(t_4)\big] r(PQ). \end{aligned} \tag{4.28}$$

(ii) The expansion formula for the power of M in (4.22) can be decomposed as

$$\begin{aligned} M^k &= t_1^k L_1 + t_2^k L_2 + t_3^k L_3 + t_4^k L_4 \\ &= t_1^k I_m + \big(t_3^k - t_1^k\big) P + \big(t_2^k - t_1^k\big) Q + \big(t_1^k - t_2^k - t_3^k + t_4^k\big) PQ \end{aligned} \tag{4.29}$$

for any integer $k \geqslant 2$.

(iii) The expansion formula for the exponential of M in (4.22) can be decomposed as

$$\begin{aligned} e^M &= e^{t_1} L_1 + e^{t_2} L_2 + e^{t_3} L_3 + e^{t_4} L_4 \\ &= e^{t_1} I_m + \big(e^{t_3} - e^{t_1}\big) P + \big(e^{t_2} - e^{t_1}\big) Q + \big(e^{t_1} - e^{t_2} - e^{t_3} + e^{t_4}\big) PQ. \end{aligned} \tag{4.30}$$

(iv) If $t_1 t_2 t_3 t_4 \neq 0$, then M in (4.22) is nonsingular too, and the inverse of the M can be written as

$$\begin{aligned} M^{-1} &= t_1^{-1} L_1 + t_2^{-1} L_2 + t_3^{-1} L_3 + t_4^{-1} L_4 \\ &= t_1^{-1} I_m + \big(t_3^{-1} - t_1^{-1}\big) P + \big(t_2^{-1} - t_1^{-1}\big) Q + \big(t_1^{-1} - t_2^{-1} - t_3^{-1} + t_4^{-1}\big) PQ. \end{aligned} \tag{4.31}$$

(v) If $t_1 t_2 t_3 t_4 = 0$, then the Moore–Penrose inverse of M in (4.22) can be decomposed as

$$\begin{aligned} M^{\dagger} &= t_1^{\dagger} L_1 + t_2^{\dagger} L_2 + t_3^{\dagger} L_3 + t_4^{\dagger} L_4 \\ &= t_1^{\dagger} I_m + \big(t_3^{\dagger} - t_1^{\dagger}\big) P + \big(t_2^{\dagger} - t_1^{\dagger}\big) Q + \big(t_1^{\dagger} - t_2^{\dagger} - t_3^{\dagger} + t_4^{\dagger}\big) PQ. \end{aligned} \tag{4.32}$$

In particular, the DOPD of the matrix pencil $aP + aQ$ under $PQ = QP$ is

$$aP + aQ = a(P - PQ) + b(Q - PQ) + (a+b)PQ. \tag{4.33}$$

Hence, a, b and $a+b$ are eigenvalues of $aP + aQ$, and the following expansions hold

$$i_{\pm}(aP + bQ) = i_{\pm}(a)r(P) + i_{\pm}(b)r(Q) + \big[i_{\pm}(a+b) - i_{\pm}(a) - i_{\pm}(b)\big]r(PQ), \tag{4.34}$$

$$(aP + bQ)^k = a^k P + b^k Q + \big[(a+b)^k - a^k - b^k\big]PQ, \tag{4.35}$$

$$e^{aP+bQ} = e^a P + e^b Q + \big(e^{a+b} - e^a - e^b\big)PQ, \tag{4.36}$$

$$(aP + bQ)^{\dagger} = a^{\dagger} P + b^{\dagger} Q + \big[(a+b)^{\dagger} - a^{\dagger} - b^{\dagger}\big]PQ. \tag{4.37}$$

In addition, general solutions to the idempotent, tripotent and involutory equations $M^2 = M$, $M^2 = I_m$ and $M^3 = M$ can also be derived from (4.25) and (4.27). Some previous and recent work on idempotency, tripotency and involution of linear combinations of idempotent matrices can be found, e.g., in [3,4,7,40,53].

Without much effort, the above results on a pair of commutative orthogonal projectors can be extended to a triple or more mutually commutative orthogonal projectors. For instance, if a triple orthogonal projectors $P_1, P_2, P_3 \in \mathbb{C}^m_{\mathrm{OP}}$ satisfy $P_iP_j = P_jP_i$, $i = 1, 2, 3$, then the corresponding eight-term matrix pencil

$$M = a_0 I_m + a_1 P_1 + a_2 P_2 + a_3 P_3 + a_{12} P_1 P_2 + a_{13} P_1 P_3 + a_{23} P_2 P_3 + a_{123} P_1 P_2 P_3 \tag{4.38}$$

is Hermitian as well, where $a_0, a_1, a_2, a_3, a_{12}, a_{13}, a_{23}, a_{123}$ are real numbers, and all the matrix pencils generate a commutative algebra up to eight dimensions over the real number field under the conventional addition and multiplication of matrices. A USFE associated with the pencil and the corresponding DOPD were given in [53], which can be used to produce various expansion formulas for the inertia, rank, power, exponential, inverse and Moore–Penrose inverse of M in (4.38).

Finally, we give an application of Theorem 3.2 to a $k \times k$ block Hermitian matrices consisting of orthogonal projectors.

Theorem 4.3. *Let $P, P_1, \ldots, P_k \in \mathbb{C}^m_{\mathrm{OP}}$, and denote*

$$M = \begin{bmatrix} P - kP_1 & P & \cdots & P \\ P & P - kP_2 & \cdots & P \\ \vdots & \vdots & \ddots & \vdots \\ P & P & \cdots & P - kP_k \end{bmatrix}, \qquad N = \begin{bmatrix} P & P_1 & 0 & \cdots & 0 \\ P & 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P & 0 & 0 & \cdots & P_k \end{bmatrix}. \tag{4.39}$$

Then,

$$i_{+}(M) = r(N) - r(P_1) - \cdots - r(P_k), \tag{4.40}$$

$$i_{-}(M) = r(N) - r(P), \tag{4.41}$$

$$r(M) = 2r(N) - r(P) - r(P_1) - \cdots - r(P_k), \tag{4.42}$$

$$s(M) = r(P) - r(P_1) - \cdots - r(P_k). \tag{4.43}$$

Proof. It is easily verified that both

$$\frac{1}{k}\begin{bmatrix} P & P & \cdots & P \\ P & P & \cdots & P \\ \vdots & \vdots & \ddots & \vdots \\ P & P & \cdots & P \end{bmatrix}, \qquad \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_k \end{bmatrix}$$

are orthogonal projectors. Applying Theorem 3.2 to the difference of the two orthogonal projectors and simplifying, we obtain (4.40)–(4.43). □

5. Expansion formulas for inertias of orthogonal projectors onto ranges of block matrices

For the simplest row black matrix $M = [A, B]$, the orthogonal projectors onto its range can be represented as

$$P_M = [A, B][A, B]^{\dagger}. \tag{5.1}$$

Hence, any formula for $[A, B]^{\dagger}$ can be used to produce certain expression of P_M. Also, note that the column space of M is jointly spanned by the columns of A and B. Hence, the orthogonal projector P_M and the two orthogonal projectors P_A and P_B have some close links. One of the main concerns about (5.1) is to give its possible expansions or decompositions under various assumptions. Some previous work on this topic can be found, e.g., in [42]. In this section, we use the formulas for inertias of orthogonal projectors in the previous sections to derive a group of equalities for ranks/inertias of orthogonal projectors onto ranges of partitioned matrices.

Some simple results on the relations among the three orthogonal projectors are given below.

Lemma 5.1. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, and denote $M = [A, B]$, $A_1 = E_B A$ and $B_1 = E_A B$. Then,*

$$P_M P_A = P_A P_M = P_A, \qquad P_M P_B = P_B P_M = P_B, \tag{5.2}$$

$$\mathscr{R}(M) = \mathscr{R}(P_A + P_B), \tag{5.3}$$

$$r(P_M) = r(P_A) + r(P_{B_1}) = r(P_B) + r(P_{A_1}), \tag{5.4}$$

$$\mathscr{R}(M) = \mathscr{R}[A, B_1] = \mathscr{R}(P_A + P_{B_1}), \tag{5.5}$$

$$\mathscr{R}(M) = \mathscr{R}[A_1, B] = \mathscr{R}(P_{A_1} + P_B), \tag{5.6}$$

$$P_A P_{B_1} = P_{B_1} P_A = 0, \qquad P_{A_1} P_B = P_B P_{A_1} = 0. \tag{5.7}$$

Eq. (5.2) follows from the facts that $\mathscr{R}(A) \subseteq \mathscr{R}(M)$ and $\mathscr{R}(B) \subseteq \mathscr{R}(M)$; (5.3) follows from $\mathscr{R}(P_A + P_B) \subseteq \mathscr{R}(M)$ and $r(P_A + P_B) = r[P_A, P_B] = r(M)$; (5.4) follows from (1.2); (5.5) and (5.6) follow from

$$[A, B_1] = [A, B]\begin{bmatrix} I_n & -A^\dagger B \\ 0 & I_k \end{bmatrix}, \qquad [A_1, B] = [A, B]\begin{bmatrix} I_n & 0 \\ -B^\dagger A & I_k \end{bmatrix};$$

and (5.7) follows from (1.1).

We first give some general results on the alternative expressions of P_M in (5.1).

Theorem 5.2. *Let A, B, M, A_1 and B_1 be as given in Theorem* 5.1. *Then, P_M can be represented as*

$$P_M = \left(AA^* + BB^*\right)\left(AA^* + BB^*\right)^\dagger, \tag{5.8}$$

$$P_M = (P_A + P_B)(P_A + P_B)^\dagger, \tag{5.9}$$

$$P_M = P_A + P_{B_1} = P_{A_1} + P_B. \tag{5.10}$$

Proof. Eq. (5.8) follows directly from the expansion of $M^\dagger = M^*(MM^*)^\dagger$. Eq. (5.9) follows from (5.3). Note that both $P_A + P_{B_1}$ and $P_{A_1} + P_B$ are Hermitian. Also by (5.7),

$$(P_A + P_{B_1})^2 = P_A + P_{B_1}, \qquad (P_B + P_{A_1})^2 = P_B + P_{A_1}. \tag{5.11}$$

Thus, both $P_A + P_{B_1}$ and $P_{A_1} + P_B$ are orthogonal projectors. Combining this fact with (5.5) and (5.6) yields the unconditional decompositions in (5.10). $\square$

Applying Theorem 3.2 to the triple sides in (5.10) and simplifying, we obtain the following result. The details are omitted.

Theorem 5.3. *Let A, B, M, A_1 and B_1 be as given in Theorem* 5.1. *Then,*

$$i_+(P_M - P_A) = r(P_M - P_A) = i_-(P_A - P_{B_1}) = i_-(P_{A_1} - P_{B_1}) = r(M) - r(A), \tag{5.12}$$

$$i_+(P_M - P_{B_1}) = r(P_M - P_{B_1}) = i_+(P_A - P_{B_1}) = r(A), \tag{5.13}$$

$$i_+(P_M - P_B) = r(P_M - P_B) = i_-(P_B - P_{A_1}) = i_-(P_{A_1} - P_{B_1}) = r(M) - r(B), \tag{5.14}$$

$$i_+(P_M - P_{A_1}) = r(P_M - P_{A_1}) = i_+(P_B - P_{A_1}) = r(B), \tag{5.15}$$

$$i_+(P_M - P_{A_1} - P_{B_1}) = i_+(P_A - P_{A_1}) = i_+(P_B - P_{B_1}) = r\left(A^*B\right), \tag{5.16}$$

$$i_-(P_M - P_{A_1} - P_{B_1}) = i_-(P_A - P_{A_1}) = i_-(P_B - P_{B_1}) = r(M) + r\left(A^*B\right) - r(A) - r(B). \tag{5.17}$$

Hence,

(a) $P_M \leqslant P_A \Leftrightarrow P_M = P_A \Leftrightarrow P_A \geqslant P_{B_1} \Leftrightarrow P_{A_1} \geqslant P_{B_1} \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A)$.
(b) $P_M \leqslant P_B \Leftrightarrow P_M = P_B \Leftrightarrow P_B \geqslant P_{A_1} \Leftrightarrow P_{B_1} \geqslant P_{A_1} \Leftrightarrow \mathscr{R}(A) \subseteq \mathscr{R}(B)$.
(c) $P_M \leqslant P_{A_1} + P_{B_1} \Leftrightarrow P_A \leqslant P_{A_1} \Leftrightarrow P_B \leqslant P_{B_1} \Leftrightarrow A^*B = 0$.
(d) $P_M \geqslant P_{A_1} + P_{B_1} \Leftrightarrow P_A \geqslant P_{A_1} \Leftrightarrow P_B \geqslant P_{B_1} \Leftrightarrow r(M) = r(A) + r(B) - r(A^*B)$.

Since the Moore–Penrose inverse of M in (5.1) can be written in different forms under various assumptions. Correspondingly, the P_M in (5.1) can be represented in some particular forms. For instance, it is easily derived from (1.1) that $A^*B = 0 \Leftrightarrow B^*A = 0 \Leftrightarrow A^\dagger B = 0 \Leftrightarrow B^\dagger A = 0$. A well-known result associated with this equivalence is

$$[A, B]^\dagger = \begin{bmatrix} A^\dagger \\ B^\dagger \end{bmatrix} \quad \Leftrightarrow \quad A^*B = 0. \tag{5.18}$$

Moreover, the present author showed in [48,49] that

$$r\left([A, B]^\dagger - \begin{bmatrix} A^\dagger \\ B^\dagger \end{bmatrix}\right) = r\big[BB^*A, AA^*B\big]. \tag{5.19}$$

Hence, the equivalence in (5.18) is a direct consequence of (5.19). Under (5.18), the orthogonal projector in (5.1) can be rewritten as the sum of two orthogonal projectors

$$P_M = [A, B][A, B]^\dagger = AA^\dagger + BB^\dagger = P_A + P_B. \tag{5.20}$$

Eqs. (5.19) and (5.20) prompt us to obtain the following result on the difference of both sides of (5.20).

Theorem 5.4. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, and denote $M = [A, B]$. Then,*

$$i_+(P_M - P_A - P_B) = r(M) + r\big(A^*B\big) - r(A) - r(B), \tag{5.21}$$

$$i_-(P_M - P_A - P_B) = r\big(A^*B\big), \tag{5.22}$$

$$r(P_M - P_A - P_B) = r(M) + 2r\big(A^*B\big) - r(A) - r(B), \tag{5.23}$$

$$s(P_M - P_A - P_B) = r(M) - r(A) - r(B). \tag{5.24}$$

Hence,

(a) $P_M \geqslant P_A + P_B \Leftrightarrow P_M = P_A + P_B \Leftrightarrow A^*B = 0$.
(b) $P_M \leqslant P_A + P_B$ *if and only if* $r(M) = r(A) + r(B) - r(A^*B)$.
(c) *The signature of* $P_M - P_A - P_B$ *is zero if and only if* $r(M) = r(A) + r(B)$, *i.e.*, $\mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}$.

Proof. It is obvious that $\mathscr{R}(P_A) \subseteq \mathscr{R}(P_M)$, $\mathscr{R}(P_B) \subseteq \mathscr{R}(P_M)$, and

$$r(P_M) = r(M), \qquad r(P_A) = r(A), \qquad r(P_B) = r(B), \qquad r(P_AP_B) = r\big(A^*B\big). \tag{5.25}$$

Then, applying (4.11), (4.12), (4.15) and (5.25) to $P_M - P_A - P_B$ yields (5.20)–(5.24). Results (a)–(c) follow from (5.20)–(5.24) and Lemma 1.1. □

Other formulas for the ranks/inertias of the orthogonal projectors onto A, B and $[A, B]$ are given below.

Theorem 5.5. *Let A, B, and M be as given in Theorem 5.1. Also, assume that C is a matrix such that $\mathscr{R}(C) = \mathscr{R}(A) \cap \mathscr{R}(B)$. Then,*

$$i_+(2P_M - P_A - P_B) = r(2P_M - P_A - P_B) = 2r(M) - r(A) - r(B), \tag{5.26}$$

$$i_+(P_M - P_C) = r(P_M - P_C) = 2r(M) - r(A) - r(B). \tag{5.27}$$

Hence,

$$2P_M \leqslant P_A + P_B \quad \Leftrightarrow \quad 2P_M = P_A + P_B \quad \Leftrightarrow \quad P_M \leqslant P_C \quad \Leftrightarrow \quad P_M = P_C \quad \Leftrightarrow \quad \mathscr{R}(A) = \mathscr{R}(B). \tag{5.28}$$

Proof. It follows from Theorems 3.2 and 4.1(b). □

The following results can easily be derived from Theorem 3.2, and the details are also omitted.

Theorem 5.6. *Let $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{l\times n}$, and denote*

$$M = \begin{bmatrix} A \\ C \end{bmatrix}, \qquad N = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}.$$

Then,

$$i_-(P_M - P_N) = r(P_M - P_N) = r(A) + r(C) - r(M). \tag{5.29}$$

Hence,

$$P_M \geqslant P_N \quad \Leftrightarrow \quad P_M = P_N \quad \Leftrightarrow \quad r(M) = r(A) + r(C). \tag{5.30}$$

Theorem 5.7. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, and $D \in \mathbb{C}^{l\times k}$, and denote*

$$M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}, \qquad N = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}.$$

Then,

$$i_+(P_M - P_N) = r[A, B] - r(A), \tag{5.31}$$

$$i_-(P_M - P_N) = r[A, B] + r(D) - r(M), \tag{5.32}$$

$$r(P_M - P_N) = 2r[A, B] + r(D) - r(M) - r(A), \tag{5.33}$$

$$s(P_M - P_N) = r(M) - r(A) - r(D). \tag{5.34}$$

Hence,

(a) $P_M - P_N \geqslant 0$ *if and only if* $r(M) = r[A, B] + r(D)$.
(b) $P_M - P_N \leqslant 0$ *if and only if* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$.
(c) $P_M = P_N \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $r(M) = r(A) + r(D)$.
(d) *The signature of* $P_M - P_N$ *is zero if and only if* $r(M) = r(A) + r(D)$.

Theorem 5.8. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times k}$, and denote*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad N = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}.$$

Then,

$$i_+(P_M - P_N) = r[A, B] + r[C, D] - r(A) - r(D), \tag{5.35}$$

$$i_-(P_M - P_N) = r[A, B] + r[C, D] - r(M), \tag{5.36}$$

$$r(P_M - P_N) = 2r[A, B] + 2r[C, D] - r(M) - r(A) - r(D), \tag{5.37}$$

$$s(P_M - P_N) = r(M) - r(A) - r(D). \tag{5.38}$$

Hence,

(a) $P_M - P_N \geqslant 0$ *if and only if* $r(M) = r[A, B] + r[C, D]$.
(b) $P_M - P_N \leqslant 0 \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $\mathscr{R}(C) \subseteq \mathscr{R}(D)$.
(c) $P_M = P_N$ *if and only if* $r(M) = r(A) + r(D)$, $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $\mathscr{R}(C) \subseteq \mathscr{R}(D)$.
(d) *The signature of* $P_M - P_N$ *is zero if and only if* $r(M) = r(A) + r(D)$.
(e) *If* $M \geqslant 0$, *then* $P_M \leqslant P_N$.

More expressions consisting of the orthogonal projectors onto the block matrix $M = [A, B]$ and its submatrices can be formulated, such as,

$$P_M - 2^{-1}P_AP_B - 2^{-1}P_BP_A, \qquad P_M - P_A - P_B + 2^{-1}P_AP_B + 2^{-1}P_BP_A,$$
$$P_M - 2P_A(P_A + P_B)^{\dagger}P_B, \qquad P_M - P_A - P_B + 2P_A(P_A + P_B)^{\dagger}P_B.$$

It is also of interest to derive possible closed-form formulas for the ranks/inertias of these matrix expressions.

6. Expansion formulas for inertias of Hermitian unitary matrices and their operations

It is easy to verify that if $A \in \mathbb{C}^m_{\mathrm{HU}}$, then its transformations $P = (I_m \pm A)/2$ satisfy $P^2 = P = P^*$, namely, the two matrices P are orthogonal projectors. In view of this fact, the formulas for inertias of orthogonal projectors in the previous sections can be extended to Hermitian unitary matrices through the transformations. For instance, if both A and B are Hermitian unitary matrices of the same size, then their transformations $P = (I_m \pm A)/2$ and $Q = (I_m \pm B)/2$ are orthogonal projectors, and the matrix pencil $\lambda I_m + \lambda_1 A + \lambda_2 B$ can be represented as

$$\lambda I_m + \lambda_1 A + \lambda_2 B = (\lambda - \lambda_1 - \lambda_2)I_m + 2\lambda_1 P_1 + 2\lambda_2 Q_1, \tag{6.1}$$

$$\lambda I_m + \lambda_1 A + \lambda_2 B = (\lambda + \lambda_1 + \lambda_2)I_m - 2\lambda_1 P_2 - 2\lambda_2 Q_2, \tag{6.2}$$

where

$$P_1 = (I_m + A)/2, \qquad Q_1 = (I_m + B)/2, \qquad P_2 = (I_m - A)/2, \qquad Q_2 = (I_m - B)/2$$

are four orthogonal projectors. In particular, we have

$$A + B = 2I_m - 2P_1 - 2Q_1 = -2I_m + 2P_2 + 2Q_2. \tag{6.3}$$

Applying Corollary 4.2 to (6.3) yields the following result.

Theorem 6.1. *Let $A, B \in \mathbb{C}^m_{\mathrm{HU}}$. Then,*

$$i_+(A+B) = r\big[(I_m + A)(I_m + B)\big] = 2^{-1}\operatorname{tr}(A+B) + r\big[(I_m - A)(I_m - B)\big], \tag{6.4}$$

$$i_-(A+B) = -2^{-1}\operatorname{tr}(A+B) + r\big[(I_m + A)(I_m + B)\big] = r\big[(I_m - A)(I_m - B)\big], \tag{6.5}$$

$$r(A+B) = r\big[(I_m + A)(I_m + B)\big] + r\big[(I_m - A)(I_m - B)\big], \tag{6.6}$$

$$s(A+B) = 2^{-1}\operatorname{tr}(A+B). \tag{6.7}$$

Hence,

(a) $A + B > 0$ *if and only if* $A = B = I_m$.
(b) $A + B < 0$ *if only if* $A = B = -I_m$.
(c) $A + B \geqslant 0$ *if and only if* $(I_m - A)(I_m - B) = 0$.
(d) $A + B \leqslant 0$ *if and only if* $(I_m + A)(I_m + B) = 0$.

Proof. Applying (4.20) to (6.1) and simplifying, we obtain

$$\begin{aligned} i_\pm(\lambda_1 A + \lambda_2 B) &= i_\pm\big[-(\lambda_1+\lambda_2)I_m + 2\lambda_1 P_1 + 2\lambda_2 Q_1\big] \\ &= i_\pm\big[-(\lambda_1+\lambda_2)I_m\big] - i_\mp(2\lambda_1 P_1) - i_\mp(2\lambda_2 Q_1) \\ &\quad + i_\mp\begin{bmatrix} [(2\lambda_1)^{-1} - (\lambda_1+\lambda_2)^{-1}]P_1 & -(\lambda_1+\lambda_2)^{-1}P_1Q_1 \\ -(\lambda_1+\lambda_2)^{-1}Q_1P_1 & [(2\lambda_2)^{-1} - (\lambda_1+\lambda_2)^{-1}]Q_1 \end{bmatrix} \\ &= i_\mp\big[(\lambda_1+\lambda_2)I_m\big] - i_\mp\big[\lambda_1(I_m + A)\big] - i_\mp\big[\lambda_2(I_m + B)\big] \\ &\quad + i_\mp\begin{bmatrix} [(2\lambda_1)^{-1} - (\lambda_1+\lambda_2)^{-1}](I_m + A) & -(\lambda_1+\lambda_2)^{-1}(I_m + A)(I_m + B) \\ -(\lambda_1+\lambda_2)^{-1}(I_m + B)(I_m + A) & [(2\lambda_2)^{-1} - (\lambda_1+\lambda_2)^{-1}](I_m + B) \end{bmatrix}. \end{aligned} \tag{6.8}$$

Applying (4.20) to (6.2) and simplifying, we obtain

$$\begin{aligned} i_\pm(\lambda_1 A + \lambda_2 B) &= i_\pm\big[(\lambda_1+\lambda_2)I_m\big] - i_\pm\big[\lambda_1(I_m - A)\big] - i_\pm\big[\lambda_2(I_m - B)\big] \\ &\quad + i_\mp\begin{bmatrix} [(\lambda_1+\lambda_2)^{-1} - (2\lambda_1)^{-1}](I_m - A) & (\lambda_1+\lambda_2)^{-1}(I_m - A)(I_m - B) \\ (\lambda_1+\lambda_2)^{-1}(I_m - B)(I_m - A) & [(\lambda_1+\lambda_2)^{-1} - (2\lambda_2)^{-1}](I_m - B) \end{bmatrix}. \end{aligned} \tag{6.9}$$

Setting $\lambda_1 = \lambda_2 = 1$ in (6.8) and (6.9) leads to

$$i_\pm(A+B) = i_\mp(I_m) - i_\mp(I_m + A) - i_\mp(I_m + B) + r\big[(I_m + A)(I_m + B)\big], \tag{6.10}$$

$$i_\pm(A+B) = i_\pm(I_m) - i_\pm(I_m - A) - i_\pm(I_m - B) + r\big[(I_m - A)(I_m - B)\big]. \tag{6.11}$$

Also, note that

$$\begin{aligned} &i_+(I_m) = m, \qquad i_-(I_m) = 0, \\ &i_+(I_m + A) = 2^{-1}\operatorname{tr}(I_m + A), \qquad i_-(I_m + A) = 0, \qquad i_+(I_m - A) = 2^{-1}\operatorname{tr}(I_m - A), \qquad i_-(I_m - A) = 0, \\ &i_+(I_m + B) = 2^{-1}\operatorname{tr}(I_m + B), \qquad i_-(I_m + B) = 0, \qquad i_+(I_m - B) = 2^{-1}\operatorname{tr}(I_m - B), \qquad i_-(I_m - B) = 0. \end{aligned}$$

Therefore, (6.10) and (6.11) reduce to (6.4) and (6.5). Eqs. (6.6) and (6.7) follow from (6.4) and (6.5). Results (a)–(d) follow (6.4) and (6.5) and Lemma 1.1. □

More formulas for the partial inertias of two Hermitian unitary matrices and their operations can be derived. For instance, applying Theorem 2.1 to the sum of two Hermitian unitary matrices $A, B \in \mathbb{C}^{m\times m}$ leads to

$$i_\pm\big[(A+B) - (A+B)^2\big] = i_\pm(A+B) + i_\pm(I_m - A - B) - i_\pm(I_m), \tag{6.12}$$

$$i_\pm\big[I_m - (A+B)^2\big] = i_\pm(I_m + A + B) + i_\pm(I_m - A - B) - i_\pm(I_m), \tag{6.13}$$

$$\begin{aligned} i_\pm(AB + BA) &= i_\mp\big[I_m - (A/\sqrt{2} + B/\sqrt{2})^2\big] \\ &= i_\mp(\sqrt{2}I_m + A + B) + i_\mp(\sqrt{2}I_m - A - B) - i_\mp(I_m). \end{aligned} \tag{6.14}$$

Note that $I_m \pm (A+B)$ and $\sqrt{2}I_m \pm (A+B)$ can be rewritten as

$$I_m + A + B = 3I_m - 2P_1 - 2Q_1 = -I_m + 2P_2 + 2Q_2, \tag{6.15}$$

$$I_m - A - B = I_m + 2P_1 + 2Q_1 = -3I_m - 2P_2 - 2Q_2, \tag{6.16}$$

$$\sqrt{2}I_m + A + B = (\sqrt{2}+2)I_m - 2P_1 - 2Q_1 = (\sqrt{2}-2)I_m + 2P_2 + 2Q_2, \tag{6.17}$$

$$\sqrt{2}I_m - A - B = (\sqrt{2}-2)I_m + 2P_1 + 2Q_1 = (\sqrt{2}+2)I_m - 2P_2 - 2Q_2, \tag{6.18}$$

where $P_1 = (I_m + A)/2$, $Q_1 = (I_m + B)/2$, $P_2 = (I_m - A)/2$ and $Q_2 = (I_m - B)/2$ are orthogonal projectors. In these cases, applying Corollary 4.2 to (6.15)–(6.18) may yield some expansion formulas for the ranks/inertias of $I_m \pm (A+B)$ and $\sqrt{2}I_m \pm (A+B)$. Substituting these formulas into (6.12)–(6.14) may also yield some expansion formulas for the ranks/inertias of $(A+B)-(A+B)^2$, $I_m - (A+B)^2$ and $AB + BA$.

7. Concluding remarks

In this paper, we constructed some congruence transformations for block Hermitian matrices consisting of Hermitian matrices, orthogonal projectors and their operations. From these Hermitian congruence transformations and the well-known Sylvester's law of inertia, we obtained a variety of explicit expansion formulas for ranks/inertias of Hermitian matrix polynomials, orthogonal projectors, Hermitian unitary matrices and their operations. Using these formulas, we further characterized many equalities and inequalities for Hermitian matrix polynomials and orthogonal projectors in the Löwner partial ordering. The algebraic methods adopted in the manipulations are quite elementary, and the results obtained seem quite simple and interesting. Therefore, the investigation in this paper would bring us deeper understanding to properties of Hermitian matrix polynomials, orthogonal projectors, Hermitian unitary matrices and their operations.

In addition to the simple matrix expressions considered in the previous sections, various general expressions consisting of orthogonal projectors may occur in matrix theory and applications. These expressions can, in general, be written as

$$p(P_1, P_2, \ldots, P_k), \tag{7.1}$$

where $P_1, P_2, \ldots, P_k$ are a group of orthogonal projectors of appropriate sizes. If the expression is Hermitian, it would also be of interest to establish some $*$-congruent block transformation equalities associated with (7.1), and then to establish expansion formulas for the inertia/rank of the matrix expression. In particular, motivated by (3.12), (3.13), (4.3) and (4.4), a challenging task is to establish expansion formulas for the ranks/inertias of some general Hermitian matrix pencils, such as, $\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$, where $P_1, P_2, P_3 \in \mathbb{C}^m_{\mathrm{OP}}$, and $\lambda_1, \lambda_2, \lambda_3$ are nonzero real numbers, as well as, $\lambda P + \lambda_1 P_1 + \cdots + \lambda_k P_k$, where $P, P_i \in \mathbb{C}^m_{\mathrm{OP}}$ satisfy $\mathscr{R}(P_i) \subseteq \mathscr{R}(P)$, $i = 1, \ldots, k$, and $\lambda, \lambda_1, \ldots, \lambda_k$ are nonzero real numbers.

Also, we point out that the congruence transformations in the previous sections for Hermitian matrices and orthogonal projectors also hold in general frames, such as, Hermitian operators and orthogonal projectors in a Hilbert space. In this event, the inertias of self-adjoint operators can accordingly be defined; see, e.g., [18,32]. In addition, orthogonal projectors can also be defined over rings with involution through the products of elements with their Moore–Penrose inverses; see [41]. In this case, it would be of interest to consider extensions of the work in this paper to orthogonal projectors over rings with involution.

As is known to all, the rank and inertia of a matrix are two basic concepts in elementary linear algebra. Any results on ranks/inertias of matrices, in particular, various closed-form formulas for ranks/inertias of matrices, are easy to understand within the scope of common knowledge in linear algebra. The conventional tools for handling ranks/inertias of matrices symbolically, as demonstrated in deriving the formulas in the previous sections, are nothing but the usual elementary operations and congruence transformations for matrices. In the past two decades, the present author has been devoting on this topic, and has proved a huge amount of results on ranks/inertias of matrices and their applications by the elementary methods mentioned above. It is expected that more and more results on ranks/inertias of matrices can be discovered, which, I believe, will become a part of core contents in linear algebra.

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References

[1] T.W. Anderson, G.P.H. Styan, Cochran's theorem, rank additivity and tripotent matrices, in: G. Kallianpur, P.R. Krishnaiah, J.K. Ghosh (Eds.), Statistics and Probability: Essays in Honor of C.R. Rao, North-Holland, 1982, pp. 1–23.
[2] Y.-H. Au-Yeung, On the semi-definiteness of the real pencil of two Hermitian matrices, Linear Algebra Appl. 10 (1975) 71–76.
[3] J.K. Baksalary, O.M. Baksalary, Idempotency of linear combinations of two idempotent matrices, Linear Algebra Appl. 321 (2000) 3–7.
[4] J.K. Baksalary, O.M. Baksalary, When is a linear combination of two idempotent matrices the group involutory matrix?, Linear Multilinear Algebra 54 (2006) 429–435.
[5] J.K. Baksalary, O.M. Baksalary, H. Özdemir, A note on linear combinations of commuting tripotent matrices, Linear Algebra Appl. 388 (2004) 45–51.

[6] J.K. Baksalary, O.M. Baksalary, T. Szulc, A property of orthogonal projectors, Linear Algebra Appl. 354 (2002) 35–39.
[7] O.M. Baksalary, Idempotency of linear combinations of three idempotent matrices, two of which are disjoint, Linear Algebra Appl. 388 (2004) 67–78.
[8] O.M. Baksalary, J. Beńitez, Idempotency of linear combinations of three idempotent matrices, two of which are commuting, Linear Algebra Appl. 424 (2007) 320–337.
[9] W. Barrett, H.T. Hall, R. Loewy, The inverse inertia problem for graphs: cut vertices, trees, and a counterexample, Linear Algebra Appl. 431 (2009) 1147–1191.
[10] J. Benítez, V. Rakočević, Applications of CS decomposition in linear combinations of two orthogonal projectors, Appl. Math. Comput. 203 (2008) 761–769.
[11] J. Benítez, V. Rakočević, On the spectrum of linear combinations of two projections in C^*-algebras, Linear Multilinear Algebra 58 (2010) 673–679.
[12] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, second ed., Springer, New York, 2003.
[13] D.S. Bernstein, Matrix Mathematics: Theory, Facts and Formulas, second ed., Princeton University Press, Princeton, 2009.
[14] J. Bérubé, R.E. Hartwig, G.P.H. Styan, On canonical correlations and the degrees of non-orthogonality in the three-way layout, in: K. Matusita, M.L. Puri, T. Hayakawa (Eds.), Statistical Sciences and Data Analysis: Proceedings of the Third Pacific Area Statistical Conference, Makuhari (Chiba, Tokyo), Japan, December 11–13, 1991, VSP International Science Publishers, Utrecht, The Netherlands, 1993, pp. 247–252.
[15] P. Binding, The inertia of a Hermitian pencil, Linear Algebra Appl. 63 (1984) 179–191.
[16] A. Björck, Numerical Methods for Least Squares Problems, SIAM, Philadelphia, PA, 1996.
[17] P.E. Bjørstad, J. Mandel, On the spectra of sums of orthogonal projections with applications to parallel computing, BIT 31 (1991) 76–88.
[18] B.E. Cain, An inertia theory for operators on a Hilbert space, J. Math. Anal. Appl. 41 (1973) 97–114.
[19] C.R. Crawford, Y.S. Moon, Finding a positive definite linear combination of two Hermitian matrices, Linear Algebra Appl. 51 (1983) 37–48.
[20] D. Hershkowitz, H. Schneider, On the inertia of intervals of matrices, SIAM J. Matrix Anal. Appl. 11 (1990) 565–574.
[21] A. Galántai, Projectors and Projection Methods, Kluwer Academic Publishers, 2004.
[22] J. Groß, On the product of orthogonal projectors, Linear Algebra Appl. 289 (1999) 141–150.
[23] J. Groß, G. Trenkler, Problem 96.4.3: orthogonal projectors, Econometric Theory 12 (1996) 744; solved by S. Puntanen, G.P.H. Styan, Econometric Theory 13 (1997) 764–765.
[24] G.H. Golub, C.F. Van Loan, Matrix Computations, third ed., Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 1996.
[25] E.V. Haynsworth, Determination of the inertia of a partitioned Hermitian matrix, Linear Algebra Appl. 1 (1968) 73–81.
[26] E.V. Haynsworth, A.M. Ostrowski, On the inertia of some classes of partitioned matrices, Linear Algebra Appl. 1 (1968) 299–316.
[27] T.M. Hoang, T. Thierauf, The complexity of the inertia, Lecture Notes in Comput. Sci. 2556 (2002) 206–217.
[28] T.M. Hoang, T. Thierauf, The complexity of the inertia and some closure properties of GapL, in: Proceedings of the Twentieth Annual IEEE Conference on Computational Complexity, 2005, pp. 28–37.
[29] L. Hogben, Handbook of Linear Algebra, Chapman & Hall/CRC, 2007.
[30] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
[31] Kh.D. Ikramov, Matrix pencils: theory, applications, and numerical methods, J. Math. Sci. 64 (1993) 783–853.
[32] C.R. Johnson, M. Lundquist, Operator matrices with chordal inverse patterns, Oper. Theory Adv. Appl. 59 (1992) 234–251.
[33] H.Th. Jongen, T. Möbert, J. Rückmann, K. Tammer, On inertia and Schur complement in optimization, Linear Algebra Appl. 95 (1987) 97–109.
[34] N.Ya. Krupnik, A.S. Markus, I.A. Fel'dman, Norm of a linear combination of projectors in Hilbert space, Funct. Anal. Appl. 23 (1989) 327–329.
[35] K. Löwner, Über monotone matrix funktionen, Math. Z. 38 (1934) 177–216.
[36] G. Marsaglia, G.P.H. Styan, Equalities and inequalities for ranks of matrices, Linear Multilinear Algebra 2 (1974) 269–292.
[37] A.W. Marshall, I. Olkin, Inequalities: Theory of Majorization and Its Applications, Academic Press, Orlando, FL, 1979.
[38] L. Mirsky, An Introduction to Linear Algebra, second corrected reprint edition, Dover, New York, 1990.
[39] Y. Nakamura, Any Hermitian matrix is a linear combination of four projections, Linear Algebra Appl. 61 (1984) 133–139.
[40] H. Özdemir, A.Y. Özban, On idempotency of linear combinations of idempotent matrices, Appl. Math. Comput. 159 (2004) 439–448.
[41] P. Patricio, C.M. Araújo, Moore–Penrose invertibility in involutory rings: the case $aa^\dagger = bb^\dagger$, Linear Multilinear Algebra 58 (2010) 445–452.
[42] C.R. Rao, H. Yanai, General definition and decomposition of projectors and some applications to statistical problems, J. Statist. Plann. Inference 3 (1979) 1–17.
[43] I.M. Spitkovsky, Once more on algebras generated by two projections, Linear Algebra Appl. 208/209 (1994) 377–395.
[44] I.M. Spitkovsky, On polynomials in two projections, Electron. J. Linear Algebra 15 (2006) 154–158.
[45] J.J. Sylvester, A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares, Lond. Edinb. Dublin Philos. Magazine J. Sci., Fourth Ser. 4 (1852) 138–142; reprinted in: H.F. Baker (Ed.), The Collected Mathematical Papers of James Joseph Sylvester, I, Cambridge University Press, 1904, pp. 378–381.
[46] R.C. Thompson, Pencils of complex and real symmetric and skew matrices, Linear Algebra Appl. 147 (1991) 323–371.
[47] R.C. Thompson, Root spreads for polynomials and Hermitian matrix pencils, Linear Algebra Appl. 220 (1995) 419–433.
[48] Y. Tian, Rank equalities for block matrices and their Moore–Penrose inverses, Houston J. Math. 30 (2004) 483–510.
[49] Y. Tian, Special forms of generalized inverses of row block matrices, Electron. J. Linear Algebra 13 (2005) 249–261.
[50] Y. Tian, Equalities and inequalities for inertias of Hermitian matrices with applications, Linear Algebra Appl. 433 (2010) 263–296.
[51] Y. Tian, Rank and inertia of submatrices of the Moore–Penrose inverse of a Hermitian matrix, Electron. J. Linear Algebra 20 (2010) 226–240.
[52] Y. Tian, Equalities for orthogonal projectors and their operations, Cent. Eur. J. Math. 8 (2010) 855–870.
[53] Y. Tian, Two universal similarity factorization equalities for commutative involutory and idempotent matrices and their applications, Linear Multilinear Algebra, in press.
[54] Y. Tian, Completing block Hermitian matrices with maximal and minimal ranks and inertias, Electron. J. Linear Algebra, in press.
[55] Y. Tian, G.P.H. Styan, Rank equalities for idempotent and involutory matrices, Linear Algebra Appl. 335 (2001) 101–117.
[56] Y. Tian, G.P.H. Styan, How to establish universal block-matrix factorizations, Electron. J. Linear Algebra 8 (2001) 115–127.
[57] Y. Tian, G.P.H. Styan, Rank equalities for idempotent matrices with applications, J. Comput. Appl. Math. 191 (2006) 77–97.
[58] N.-K. Tsing, F. Uhlig, Inertia, numerical range, and zeros of quadratic forms for matrix pencils, SIAM J. Matrix Anal. Appl. 12 (1991) 146–159.
[59] F. Uhlig, Computing the inertias in symmetric matrix pencils, Linear Algebra Appl. 201 (1994) 199–209.
[60] H. Väliaho, Determining the inertia of a matrix pencil as a function of the parameter, Linear Algebra Appl. 106 (1988) 245–258.
[61] N. Vasilevsky, I.M. Spitkovsky, On the algebra generated by two projections, Dokl. Akad. Nauk Ukrain. SSR Ser. A 8 (1981) 10–13.
[62] S. Zlobec, An explicit form of the Moore–Penrose inverse of an arbitrary complex matrix, SIAM Rev. 12 (1970) 132–134.